NATIONAL TEST PILOT SCHOOL

Professional Course

Volume 1
Math & Physics for Flight Testers

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Table of Contents

Volume 1 – Math & Physics for Flight Testers

Chapter 1

Algebra & Trigonometry

Table of Contents

1.1 Introduction

1.1.1 Algebraic Manipulation

In algebra, only like terms may be added or subtracted but there are no such restrictions on for multiplication or division.

eg $a + a = 2a$ but $a + b = a + b$

$3a - a = 2a$ but $3a - b = 3a - b$

Similarly:

$3x + x + 4x = 8x$

$7y - 3y = 4y$

$5ab + 2ab = 7ab$

$10x^2 - 8x^2 = 2x^2$

whereas, for multiplication and division:

$a \times b = ab$ $\quad 10x \times 4y = 40xy$

$p \times q = pq$ $\quad 12x \div 3x = \frac{12x}{3x} = 4$

$r \div t = \frac{r}{t}$ $\quad 5a \times 4a = 20a^2$

1.1.2 Brackets

When strings of terms are enclosed in brackets, the bracketed expression is to be treated as an entity. For example, if $(2a + 3b)$ is to be multiplied by $3a$, then each term within the bracket must be multiplied by $3a$,

$$3a(2a+ 3b) == 6a^2 + 9ab$$

Similarly, if two brackets are multiplied together, each term in the first bracket must be multiplied by each term in the second, eg

$$(p + q + r)(x + y) = px + py + qx + qy + rx + ry$$

Some important cases are:

a. $(x + y)^2 = (x + y)(x + y) = x^2 + 2xy + y^2$
b. $(x - y)^2 = (x - y)(x - y) = x^2 - 2xy + y^2$
c. $(x + y)(x - y) = x^2 - xy + xy - y^2 = x^2 - y^2$ (difference of two squares)
d. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Now try these:

For a and b show that:

a. $x^2 + 2x - 1 + 3x - 4 + 2x^2 + 5 = 3x^2 +5x$
b. $x^2 + ax + bx + ab + a^2 - ab - ax + b^2 - bx = x^2 + a^2 +b^2$

Then add the following

c. $y^2 - a^2 + 4y^2 + ay + y^2 + ay + 3a^2 + 2y +2a^2 +5y^2 + ay$
d. $p^2 + 2pq + q^2 + p^2 - 2pq + q^2 + 2p^2 - 2q^2$
e. $\theta + \alpha + 2\alpha - \theta + 3\theta + 4\alpha$

Remove the brackets from:

a. $2(x^2 - 4x)$ Ans $2x^2 - 8x$
b. $a(2a + 3b)$ Ans $2a^2 + 3ab$
c. $a^2 (a - h)$ Ans $a^3 - a^2h$
d. $2\left(xy + \frac{3}{4}x\right)$ Ans $2xy + \frac{6x}{4}$

Simplify:

a. $m - n + 2(3m + 2n)$ Ans $7m + 3n$
b. $4(p - q) - 2(p + q)$ Ans $2p + 6q$
c. $2(x^2 + 2ax + 2a^2) - x^2 + a^2$ Ans $(x + 3a)(x + a)$
d. $2(t^3 + 1.4t^2 - 2.7t) - 4(0.5t^3 - t^2 + 1.3t)$ Ans $6.81t^2 - 10.6t$
e. $3a^2 (a - h) - a(a^2 - h^2) - 2ah(a + h)$ Ans $2a^3 - 5a^2h - ah^2$

1.1.3 Factorization

Factorization is the reverse operation to algebraic multiplication. We try to split a given product back into a pair (or more) of factors. This is often a trial and error process. Eg

a. $6x^2 + 7x + 2$ — Ans $(3x + 2)(2x + 1)$
b. $p^2 - q^2$ — Ans $(p + q)(p - q)$

1.1.4 Simplification of Fractions

The normal rules for manipulating arithmetical fractions apply equally to algebraic fractions. The main source of error comes from not treating brackets as complete entities. Eg

a. $$\frac{\frac{1}{2}\rho U^2 S C_L}{\frac{1}{2}\rho U^2 S C_D} = \frac{C_L}{C_D}$$

b. $$\frac{1}{x} + \frac{1}{y} = \frac{y + x}{xy}$$

c. $$\frac{1}{(x-2)}+\frac{3}{(x-4)}=\frac{(x-4)}{(x-2)(x-4)}+\frac{3(x-2)}{(x-2)(x-4)}$$
$$=\frac{(x-4)+3(x-2)}{(x-2)(x-4)}$$
$$=\frac{4x-10}{(x-2)(x-4)}$$
$$=\frac{2(2x-5)}{(x-2)(x-4)}$$

Note: In examples b and c, just as with numerical fractions, it is necessary to find the common denominator. This is most easily achieved by multiplying the individual denominators together.

1.1.5 Indices

$b \times b \times b \times b$ is abbreviated to b^4 (b to the power 4). "*b"* is the base and 4 is the index. In general, b^n means "*b*" multiplied by itself "*n*" times. It follows therefore that

a. $b^2 \times b^3 = (b \times b) \times (b \times b \times b) = b^{(2+3)} = b^5$

b. $b^3 \div b^2 = \dfrac{(b \times b \times b)}{(b \times b)} = b^{(3-2)} = b$

ie to *multiply* two numbers in index form, *add* the indices; to *divide* two numbers in index form, *subtract* the indices. (note: the base must be the same). thus:

c. $a^4 \times a^5 = a^9$
d. $x^3 \times x^9 = x^{12}$
e. $p^3 \div p = p^2$
f. $y^{10} \div y^2 = y^8$

and it also follows that

g. $(a^x)^y = a^{xy}$

h. $a^{-n} = \dfrac{1}{a^n}$

j. $a^{\frac{1}{n}} = \sqrt[n]{a}$

Note: A very important result is that when any number is raised to the power zero, the answer is always 1.

ie $a^0 = 1$
eg $x^0 = 1$, $4^0 = 1$, $(100)^0 = 1$, $(x^2)^0 = 1$

Try the following exercises:

1. Factorize:
   a. $a^2 + ab$
   b. $x^2 - xy$
   c. $a^2 - b^2$
   d. $2pq - 3q^2$
   e. $4x^2 - y^2$
2. Factorize:
   a. $ax - 2x - a + 2$
   b. $x^2 + 5x + 4$
   c. $x^2 - 3x - 4$
   d. $y^2 - 2y - 15$
   e. $2x^2 + 3x + 1$
3. Simplify:
   a. $\frac{ab^2x}{ax^2}$
   b. $\frac{16lmn^2}{12l^2m^2}$
   c. $\frac{x^2 - xy}{x^2 + xz}$
   d. $\frac{a^2 - ab}{ac - bc}$
   e. $\frac{r^2 - s^2}{(r + s)^2}$
4. Express as single fractions
   a. $\frac{1}{x} + \frac{4}{y}$
   b. $\frac{1}{t} + \frac{2}{3 - t}$
   c. $\frac{4}{3ab} - \frac{5}{6bc}$
   d. $\frac{5x + 4}{x^2 - 2x} - \frac{7}{x - 2}$

1.2 Solution of Equations

In general, if an equation contains only one unknown quantity, the numerical value of that unknown can be found no matter how complicated the equation may appear. To solve the equation we must obey all the rules of algebra whilst isolating the unknown quantity on one side of the equation. A simple example of this is shown in detail:

Eg Solve $3x + 4 = 10$

Subtracting 4 from both sides of the equation

$$3x = 10 - 4$$

Dividing both sides of the equation by 3 $x = \frac{6}{3}$

$$x = 2$$

Note: Any operation may be carried out on an equation as long as it is to *both sides* of the "equals" sign.

Let us try some other simple examples:

Eg 1 Solve $2x + 3 = x + 6$

Subtract 3 from both sides: $2x = x + 6 - 3 = x + 3$
Subtract x from both sides: $2x - x = 3$

$x = 3$

Eg 2 Solve $5(x + 3) = 10(x - 1)$

First, multiply out the brackets: $5x + 15 = 10x - 10$
Add 10 to both sides $5x + 25 = 10x$
Subtract $5x$ from both sides $25 = 5x$
Divide by 5: $x = 5$

Now try the following examples:

Solve
a. $7x - 3 = 25$
b. $2x + 1 = 3x - 3$
c. $3(2x + 5) = 57$
d. $5(6 - 3x) = 16 + 6x$
e. $4(5x - 8) = 4(3x + 4)$

If there are several unknown quantities we must have as many independent equations as there are unknowns in order to find their numerical values.

1.2.1 Simultaneous Linear Equations

A pair of equations containing two unknowns may be solved in a number of ways. They are called simultaneous equations and the most common methods of solution are substitution and elimination. An example of each is shown below, firstly by substitution.

Eg Solve

$$5x = 2y \tag{1.1}$$

$$3x + 6y = 3.6 \tag{1.2}$$

From equation (1) , $x = +\frac{2}{5}y$, therefore we can substitute for x in equation 2 giving:

$$3\left(\tfrac{2}{5}y\right) + 6y = 3.6$$

$$7.2y = 3.6$$

$$y = \frac{3.6}{7.2} = 0.5$$

Thus $x = \frac{2}{5}y = 0.2$

Eg by elimination:

Solve

$$3x + 4y = 18 \tag{1}$$

$$2x + 3y = 13 \tag{2}$$

Multiply equation (1.1) by 2 and (1.2) by 3 to give

$$6x + 8y = 36 \tag{1.3}$$

$$6x + 9y = 39 \tag{1.4}$$

We can now eliminate x from the equations by subtracting equation (1) from equation (2):

Hence

and by substitution in (1) or (2) for y:

$$y = 3$$

$$\therefore x = \frac{36 - 24}{6} = 2$$

Do these examples using both methods:

a. $3x + 6y = 11$
$14x - y = 3$

c. $26x + 8y = 9$
$2y - 6x = 2$

d. $9x + 14y = 5$
$12x + 21y = 7$

b. $2x - 3y = 2$
$3x + 5y = 41$

1.2.2 Quadratic Equations

These are equations in which there is only one unknown but in which the unknown is of the second degree i.e. it involves a "squared" term.

Eg $x^2 = 4$; thus $x = +2$ or -2

$x^2 - 4x = 0$; thus $x = 0$ or 4

$x^2 - 3x + 2 = 0$; thus $x = 1$ or 2

There are several methods of solving quadratic equations and there will always be two solutions. The two important methods of solution are shown below:

1.2.2.1 ***By Factorization***

If the expression can be factorized, the solutions may be obtained by equating each factor to zero.

Eg $x^2 - 3x + 2 = 0$

Factorizing: $(x - 2)(x - 1) = 0$

Hence: Either $(x - 2) = 0$ or $(x - 1) = 0$

$\therefore\ x = 2$ $\therefore\ x = 1$

1.2.2.2 ***By use of the General Formula***

The general form of a quadratic equation is:

$$ax^2 + bx + c = 0$$

The solution is then given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For example, solve $x^2 - 3x + 2 = 0$ by the formula:

$$\therefore x = \frac{3 \pm \sqrt{9-8}}{2} \qquad = \frac{3 \pm \sqrt{1}}{2} \qquad = \frac{3 \pm 1}{2}$$

Hence, either $x = \frac{3+1}{2} = 2$ or $x = \frac{3-1}{2} = 1$

Which confirms the solutions obtained by the other method.

1.2.3 Examples

Solve a. $x^2 - 6x + 5 = 0$

b. $x^2 - 5x - 7 = 0$

c. $2x^2 - x - 3 = 0$

1.3 Logarithms

1.3.1 Definitions

Consider the expression a^n i.e. a multiplied by itself n times. If we let $A == a^n$, then, provided a stays constant, the value of A will depend on the value of n, that is, there is a functional relationship between n and A. This relationship is a logarithmic one and $a^n = A$ can be rewritten as:

$$n = \log_a A \text{ (i.e. } n = \text{the logarithm of } A \text{ to the base } a)$$

We are familiar with logs to the base 10. For example, tables tell us that $\log_{10} 3 = 0.4771$, which means $10^{0.4771} = 3.0$. The log of a number to base a is the power to which a must be raised to equal the number. For example:

a. $\log_{10} 100 == 2$ $(10^2 = 100)$

b. $\log_4 2 = 0.5$ $(4^{0.5} = 4^{½} = 2)$

1.3.2 Rules Concerning Logarithms

1.3.2.1 Multiplication

$$\log_a xy = \log_a x + \log_a y$$

(To multiply two and take the antilog).

1.3.2.2 Division

$$\log_a(x/y) = \log_a x - \log_a y$$

(To divide two numbers, subtract their logs and take the antilog).

1.3.2.3 *Raising to a Power*

$$\log_a x^n = n \log_a x$$

(To raise a number to a power, multiply the log of the number by the power and take the antilog).

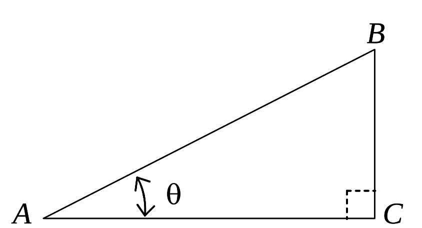
1.3.2.4 *Log Logs*

$$\log_a (\log_a x^n) = \log_a (n \log_a x)$$
$$= \log_a n + \log_a (\log_a x)$$

1.4 Trigonometry Definitions

1.4.1 Ratios of Acute Angles

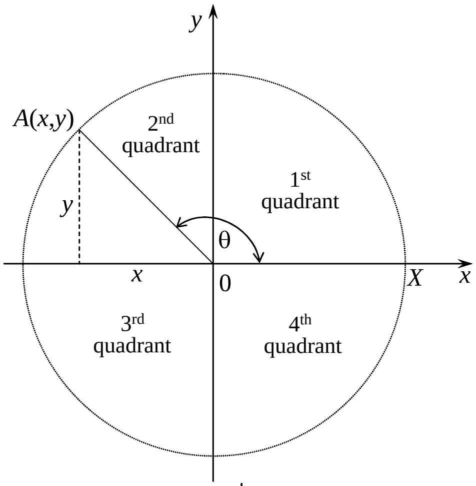
The basic trigonometrical ratios of the acute angles are defined in terms of the sides of a right-angled triangle.



$\sin\theta = BC/AB$	$\text{cosec}\,\theta = 1/\sin\theta = AB/BC$
$\cos\theta = AC/AB$	$\sec\theta = 1/\cos\theta = AB/AC$
$\tan\theta = BC/AC$ (= $\sin\theta/\cos\theta$)	$\cot\theta = 1/\tan\theta = AC/BC$

1.4.2 Ratios of Other Angles

These are defined from the following diagram.



Using $0X$ as a reference direction, the angle can be drawn (anti-clockwise positive) making a radial $0A$ where A lies on a circle of radius, say, r. The point A has co-ordinates (x,y). Then:

$$\sin\theta = y/r$$

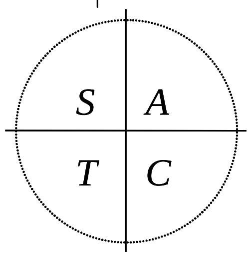
$$\cos\theta = x/r$$

$$\tan\theta = y/x$$

with the signs of x and y taking their normal positive or negative values depending on which side of the axes they are on and r always being positive

Thus:

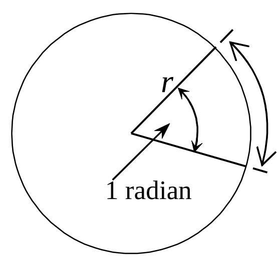
a. $\sin\theta$ is positive in the 1st and 2nd quadrants.
b. $\cos\theta$ is positive in the 1st and 4th quadrants.
c. $\tan\theta$ is positive in the 1st and 3rd quadrants.



Mnemonic showing which of the ratios are positive

1.4.3 Radian Measure

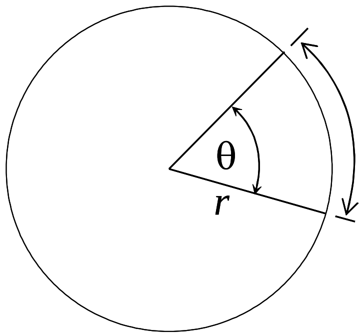
Practical measurements of angles are made in degrees and minutes, one complete revolution being divided into 360 degrees. This arbitrary number 360 is not suitable for theoretical work and the more basic unit of the radian is used. The radian is defined to be the angle subtended at the centre of a circle by an arc equal in length to the radius.



Since the circumference of a circle has a length 2π, one complete revolution is equivalent to an angle of 360° or 2π radians. The conversion factors are therefore:

a. From Degrees to "Radians: $x \text{ degrees} = \dfrac{2\pi x}{360} = \dfrac{\pi x}{180} \text{ radians}$

b. From Radians to Degrees: $y \text{ radians} = \dfrac{360y}{2\pi} = \dfrac{180y}{\pi} \text{ degrees}$



The length of a circular arc is $r\theta$ where r is the radius of the circle and θ is the angle subtended by the arc (measured in radians).

1.4.4 Important Trigonometrical Relationships

The following trigonometrical relations are stated, without proof, but they will be required in later work:

a. $\sin^2 \theta + \cos^2 \theta = 1$
b. $\sec^2 \theta = 1 + \tan^2 \theta$
c. $\sin (A + B) = \sin A \cos B + \sin B \cos A$
d. $\sin (A - B) = \sin A \cos B - \sin B \cos A$
e. $\cos (A + B) = \cos A \cos B - \sin A \sin B$
f. $\cos (A - B) = \cos A \cos B + \sin A \sin B$
g. $\sin^2 A = 2 \sin A \cos A$
h. $\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$

1.4.5 Graphical Representation of Trigonometrical Functions

By constructing tables of values of $\sin x$ and $\cos x$, the graphs of $y = \sin x$ and $y = \cos x$ may be drawn:

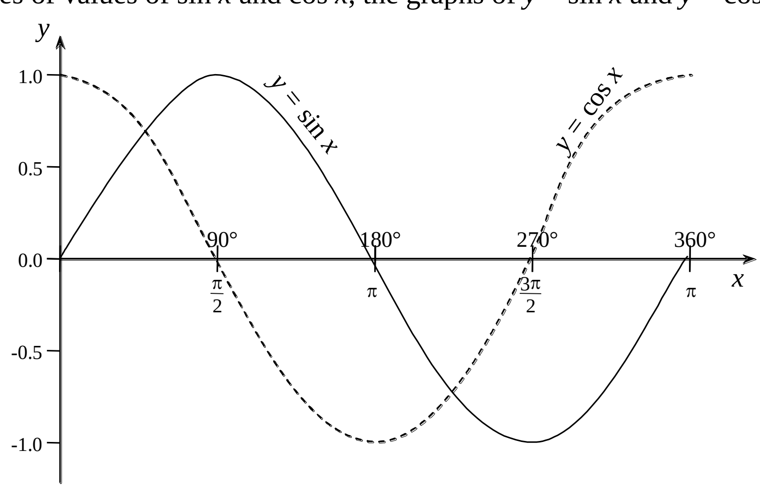


Figure 1.1 Graphs of $y = \sin x$ and $y = \cos x$

If we had drawn the graphs of $y = \sin x$ and $y = \cos x$ over a greater range of values of x, we would have found that the graphs would repeat themselves every 360° (or 2π radians). A function which repeats itself at regular intervals is called a periodic function, and the interval between two successive repetitions is called the period of the function. Thus both $\sin x$ and $\cos x$ are periodic functions of x, having periods of 360° (or 2π radians). It can also be seen from the graphs that both $\sin x$ and $\cos x$ oscillate between a maximum value of +1.0 and a minimum value of –1.0. This maximum/minimum value of the function is called the amplitude.

1.4.6 The Function *A* Sin *x*

If each ordinate of the graph of sin x is-multiplied by some constant '*A*', then the maximum/minimum values will be ± *A*, but no alteration to the period of the function will occur. Thus the number '*A*' determines the amplitude of the function.

1.4.7 The Function Sin ω*x*

Consider the graph of $y = \sin 2x$ (i.e. ω = 2). This will complete one oscillation in 180° so that the period is 180° or π radians. The graph of $y = \sin 3x$ (i.e. ω = 3) will complete one oscillation in 120° or 2π/3 radians. In general the graph of $y = \sin \omega x$ completes one oscillation in 360/ω degrees or 2π/ω radians. Its period is therefore 2π/ω, that is, the value of ω determines the period. If one oscillation is completed in an interval 2π/ω, then ω/π2 oscillations will occur in an interval of one radian along the *x*-axis. That is, the *frequency* of the oscillations is ω/π2 per radian or ω per revolution (as one revolution = 2π). Thus:

a. Frequency (per radian) = $\dfrac{\omega}{2\pi} = \dfrac{1}{period}$

b. Frequency (per revolution) = $\omega = \dfrac{2\pi}{period} =$ Angular frequency

For example:

$V = 240 \sin 100t$ describes an ac signal.

Period = $\dfrac{2\pi}{100\pi} = \dfrac{1}{50}$ second

Frequency = $\dfrac{100\pi}{2\pi} = 50Hz$

Angular frequency = 100π rad/sec

Amplitude = 240 volts.

1.4.8 Phase Angle

The graph of $y = \sin\left(x + \frac{\pi}{4}\right)$ has the same period and amplitude as $y = \sin x$ but it is displaced horizontally through π/4. It is said to lead the graph of $y = \sin x$ by π/4 since its maximum/minimum values occur π/4 before those of $y = \sin x$.

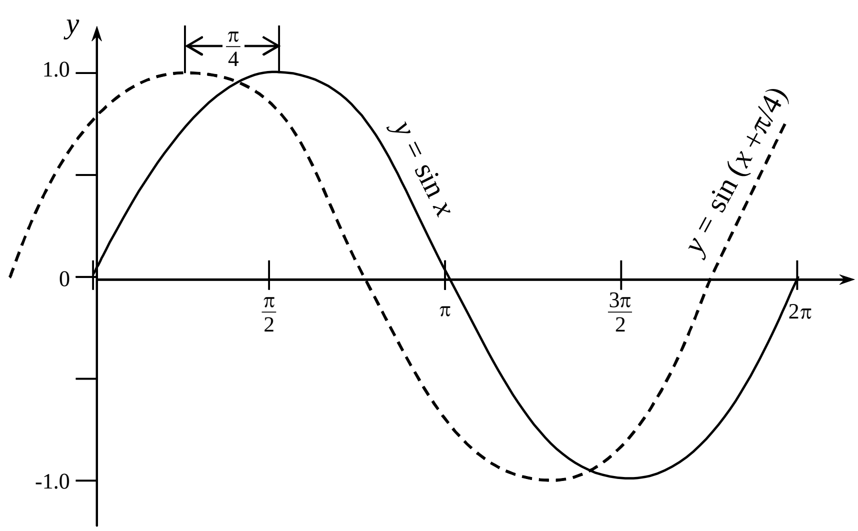


Figure 1.2 Phase Angle

1.4.9 Phase Lead/Lag

In general $y = a \sin(\omega t + \alpha)$ leads $y == a \sin \omega t$ by α. α is known as the phase lead. If α is negative then $y = a \sin(\omega t + \alpha)$ lags $y == a \sin \omega t$ and α is known as a phase lag (note this phase lead/lag is an angle as both ωt and α are angles).

1.4.10 Time Lead/Lag

Consider $y = a \sin(\omega t + \alpha)$ and $y = a \sin \omega t$ again. We have seen that in terms of angles $y = a \sin(\omega t + \alpha)$ leads $y == a \sin \omega t$ by the angle α. However in terms of time $y = a \sin(\omega t + \alpha)$ leads $y = a \sin \omega t$ by α/ω seconds.

1.4.11 Phase Difference

Consider two oscillations represented by the equations:

$$y = a \sin(\omega t + \alpha) \text{ and } y = b \sin(\omega t + \beta)$$

The phase of the first at time t is $(\omega t + \alpha)$ and the phase of the second at the same time is $(\omega t + \beta)$. The *phase difference* is the difference between these phases ie $(\alpha - \beta)$. In this case, where we are comparing the phase of the first oscillation with that of the second, the first oscillation is said to lead the second if $(\alpha - \beta)$ is positive and to lag the second if $(\alpha - \beta)$ is negative. If a = β the two oscillations have the same phase at any time and they are said to be in phase. Note that the phase difference is an angle. It is possible to write the phase difference in terms of a time by dividing $(\alpha - \beta)$ by ω.

It may be noted that since:

$$a\cos(p\theta + \beta) = a\sin\left(p\theta + \beta\frac{\pi}{2}\right)$$, the equation $y = a \cos(p\theta + \beta)$

represents an oscillation of the same amplitude (a) and period ($2\pi/p$) as $y = a \sin(p\theta + \beta)$ but leading it by $\pi/2$.

1.4.12 Combination of Sin and Cos Types of Oscillations

Consider an oscillation which is made up of two component oscillations $y == a \sin \omega x$ and $y = b \sin \omega x$. Although the amplitudes of these component oscillations are different the frequencies are the same. The final equation of the oscillation may be written as:

$$\begin{aligned} y &= a\sin \omega x + b\cos \omega x \\ &= \sqrt{a^2 + b^2}\sin(\omega x + \delta) \\ &= \sqrt{a^2 + b^2}\cos(\omega x + \varepsilon) \end{aligned}$$

where $$\tan\delta = \frac{b}{a} \qquad \tan\varepsilon = -\frac{a}{b}$$

1.5 The Exponential Function (e^x)

1.4.13 Introduction

In the physical world there are frequent occurrences of physical phenomena in which the rate of change of some quantity is proportional to the quantity itself. This is known as the law of natural growth or decay and some examples are:

a. Newton's Law of Cooling which states that the rate of decrease at any instant of the excess temperature of a body over its surroundings is proportional to that excess temperature.

b. Radio active substances decay at a rate which at any instant is proportional to the quantity of substance present.

This natural growth function is called the *exponential function*, e^x, and it may be defined mathematically in several ways. One of the simplest is. to consider "the series"

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = f(x)$$

Note: ! denotes a "factorial": eg $5! = 5 \times 4 \times 3 \times 2 \times 1$
$3! = 3 \times 2 \times 1$

The value of *y* may be found for any value of *x* by substituting that value of *x* into the series.

Eg When: $x = 0, y =$

$$x = \tfrac{1}{2}, \quad y = 1 + \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{48} + \cdots \cong 1.6487$$

$$x = 1, \quad y = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \cdots \cong 2.7183$$

$$x = 2, \quad y = 1 + 2 + 2 + \tfrac{4}{3} + \tfrac{2}{3} + \cdots \cong 7.3891$$

$$x = 3, \quad y = 1 + 3 + \tfrac{9}{2} + \tfrac{9}{4} + \tfrac{27}{3} + \cdots \cong 20.0855$$

Inspection of the results obtained shows that

$$1.6487 = (2.7183)^{½}$$

$$2.7183 = (2.7183)^{1}$$

$$1\ 7.3891 = (2.7183)^{2}$$

$$2\ 20.0855 = (2.7183)^{3}$$

and $$1 = (2.7183)^{0}$$

The number 2.7183. . . obviously has some fundamental relationship with the series and is denoted by the symbol '*e*'.

i.e. $e = 2.7183 \ldots$

Hence, when: $x = 0, \quad y = e^0$

$x = 1/2, \quad y = e^{½}$

$x = l, \quad y = e^1$

$x = 2, \quad y = e^2$

$x = 3, \quad y = e^3$

Thus, the series we started with gives: $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots= f(x)=e^x$

This can be proved by more rigorous mathematics, but for our purposes, this is sufficient.

$$e^x=1+x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots$$

and it can be shown that: $$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots$$

and $$e^{ax}=1+ax+\frac{a^2x^2}{2!}+\frac{a^3x^3}{3!}+\frac{a^4x^4}{4!}+\frac{a^5x^5}{5!}+\cdots$$

Also, e^x obeys all the normal rules of indices, eg

$$e^a\times e^b=e^{a+b}$$
$$\frac{e^a}{e^b}=e^{a-b}$$
$$\left(e^a\right)^b=e^{ab}$$
$$e^{-x}=\frac{1}{e^x}$$

1.5 Natural Logarithms

1.5.1 Definition

We have seen, from the definition of logarithms that if $A = a^n$ then $n = \log_a A$. If we use the exponential function, e^x, we can say that if $a = e^x$ then $x = \log_e a$.

Logarithms to the base '*e*' are called *natural* or *napierian logs*. Base 10 logs can be easily obtained from natural logs by use of the relation:

$$\log_{10} a=\frac{\log_e a}{\log_e 10}$$
$$=\frac{1}{2.3026}\log_e a$$
$$=0.4343\log_e a$$

Note: '$\log_e a$' is often written as '$\ln a$'; i.e. ln is an alternative notation to $\log_e$.

1.6.2 Graphs of e^x and $\log_e x$

The graph of $y = e^x$ can be plotted using the series of values previously obtained. Other exponential functions are also shown in the figure below:

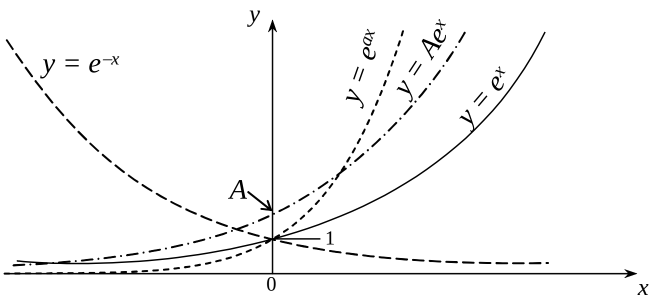


Figure 1.3 Exponential Functions

Points to notice are that:

a. If the index of e^{ax} is positive y will increase as x increases.

b. If the index of e^{ax} is negative y will decrease as x increases.

c. The value of e^{ax} (no matter what value a takes) when $x = 0$ is unity.

d. The graph of $y = Ae^{ax}$ passes through the point (0, A).

The graph of $y = \log_e x$ is shown below. It is worth noting that as x decreases y tends to minus infinity. It is not possible to obtain values of y for values of x less than zero (i.e. for negative values of x).

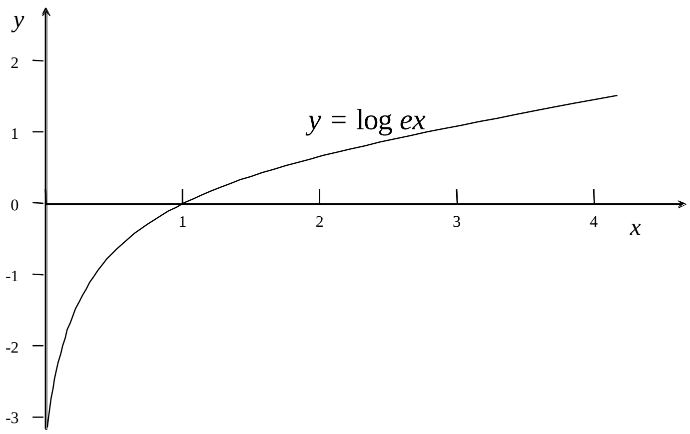


Figure 1.4 Graph of $y = \log_e x$

1.6 Relationships Between Exponential and Trigonometrical Functions

Trig functions such as sin x and cos x can be represented as series. These are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Using series for e^x, we can write out the series for e^{jx} (where $j = \sqrt{-1}$ as used in complex number theory in the pre course study material):

$$e^{jx} = 1 + jx + \frac{j^2x^2}{2!} + \frac{j^3x^3}{3!} + \frac{j^4x^4}{4!} + \cdots$$

$$e^{jx} = 1 + jx - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \frac{jx^5}{5!} - \frac{x^6}{6!} - \frac{jx^7}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} - + \frac{x^4}{4!} - - \frac{x^6}{6!} + \cdots + j\left(x - \frac{x^3}{3!} - + \frac{x^5}{5!} - - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos x + j \sin x$$

Thus $$e^{jx} = \cos x + j \sin x$$

Further, had we written out the series for e^{-jx}, we would have obtained:

$$e^{-jx} = \cos x - j \sin x$$

This gives us another relationship since, adding these expressions, we get:

$$e^{jx} + e^{-jx} = (\cos x + j\sin x) + (\cos x - j\sin x)$$

$$= 2\cos x$$

$$\therefore \cos x = \tfrac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

Also $$e^{jx} - e^{-jx} = (\cos x + j\sin x) - (\cos x - j\sin x)$$

$$= j\,2\sin x$$

$$\sin x = \tfrac{1}{j2}\left(e^{jx} - e^{-jx}\right)$$

Finally $$e^{(a+jb)x} = e^{ax} \cdot e^{jbx}$$

$$e^{(a+jb)x} = e^{ax}(\cos bx + j\sin bx)$$

and similarly $$e^{(a-jb)x} = e^{ax}(\cos bx - j\sin bx)$$

Summarizing:

a. $e^{jx} = \cos x + j\sin x$

b. $e^{-jx} = \cos x - j\sin x$

c. $\cos x = \tfrac{1}{2}\left(e^{jx} + e^{-jx}\right)$

d. $\sin x = \tfrac{1}{j2}\left(e^{jx} - e^{-jx}\right)$

e. $e^{(a\pm jb)x} = e^{ax}(\cos bx \pm j\sin bx)$

1.7 Complex Number Arithmetic

The pre-course study package has provided an introduction to complex numbers which we can now build on. As a reminder, first consider some powers of j

a. $j = \sqrt{-1}$	c. $j^3 = \left(j^2\right)j = -1j = -j$
b. $j^2 = \sqrt{-1}\cdot\sqrt{-1} = -1$	d. $j^4 = \left(j^2\right)j^2 = (-1)^2 = 1$

1.7.1 Addition and Subtraction

The rules are simply that we must add (or subtract) the real parts together and the imaginary parts together

Eg $(4 + j3) + (6 - j) = 10 + j2$

$(6 - j2) - (4 + j5) = 2 - j7$

so, in general, $(a + jb) + (c + jd) == (a + c) + j(b + d)$

1.7.2 Multiplication

This is carried out in the same way as you would determine an algebraic product of the form $(3x + 2y)(2x + 5y)$. Eg

$$(3 + j6)(7 + j2) = 21 + j42 + j6 + 12j^2$$

$$= 21 + j48 - 12$$

$$= 9 + j48$$

1.7.3 Division

To perform this operation, we must multiply the numerator and denominator by the *complex conjugate* of the latter in order to convert it to a real number.

Eg
$$\frac{(7-j4)}{(4+j3)} = \frac{(7-j4)}{(4+j3)} \cdot \frac{(4-j3)}{(4-j3)} = \frac{28-j37-12}{16+9}$$
$$= \frac{16-j37}{25} = \frac{16}{25} - j\frac{37}{25}$$
$$= 0.64 - j1.48$$

1.7.4 Equal Complex Numbers

If we know that $a + jb = c + jd$

then $a = c$

and $b = d$

ie the two real parts are equal and the two imaginary parts are equal.

Eg If $x + jy = 5 + j2$

then $x = 5$

and $y = 2$

1.7.5 Argand Diagrams

This is a graphical representation of a complex number. They are drawn as vectors on a set of axes with the *x*-axis representing the real part and the *y*-axis the imaginary part of the complex number:

0*A* is the vector representing $(5 + j2)$

0*B* is the vector representing $(-4 + j3)$

0*C* is the vector representing $(-2 - j3)$

0*D* is the vector representing $(3 - j2)$

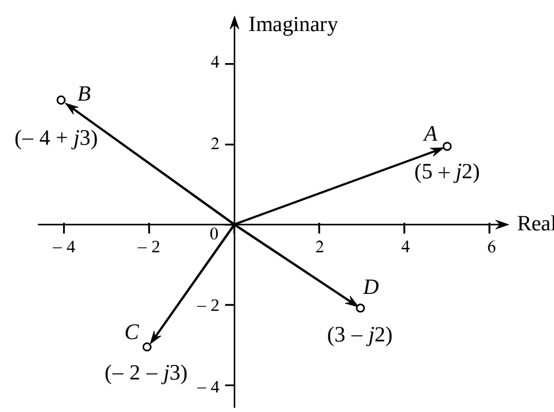


Figure 1.5 Argand Diagram

1.7.6 Polar Form of Complex Numbers

Instead of using the normal Cartesian coordinates (*x*, *y*) to represent a complex number (*x* + *jy*) on a graph we can use the Polar form or (*r*, θ) form of coordinate. The relation is best explained in Figure 2.6.

$$r^2 = x^2 + y^2 \qquad \therefore \qquad r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x} \qquad \therefore \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

So, instead of *x* + *jy*, we can write

$$(r\cos\theta + jr\sin\theta)$$

ie $x + jy = r(\cos\theta + j\sin\theta)$

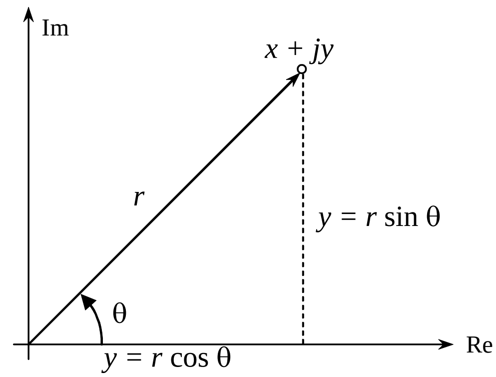


Figure 1.6 Polar Form of Complex Number

The form $r(\cos\theta + j\sin\theta)$ is the *polar form* of a complex number and is often shortened to *r*/θ. *r* is known as the *modulus* of the complex number and θ the *argument*, *r* is *always* positive and θ lies between –180° and +180° (–π rads and + π rads).

1.7.7 Exponential Form of a Complex Number

We have seen previously that

$$e^{\theta j} = (\cos\theta + j\sin\theta)$$

$$\therefore \qquad re^{j\theta} = r(\cos\theta + j\sin\theta)$$

Thus, we have written the complex number in exponential form: $re^{\theta j}$

1.7.8 Summary

To summarize, the complex number, *z* may be written

$z = a + jb$	
$z = r(\cos\theta + j\sin\theta)$	polar form
$z = r^{\theta j}$	exponential form

Note: The exponential form is obtained from the polar form, *r* is the same in both cases but θ *must* be in *radians* in exponential form.

Volume 1 – Math & Physics for Flight Testers

Chapter 2

Calculus

Table of Contents

2.1 Preface

This book on Calculus is designed as a refresher for test pilot and flight test engineer candidates prior to entering test pilot school. This pre-TPS course assumes that students are familiar with calculus, however, it has been a few years since the subject has been used and a short refresher course is required. The subject matter has been condensed to the minimum required to enter a test pilot school and all rigorous treatments have been eliminated from this text. If the students are not familiar with calculus then it is recommended that their studies be expanded to include a more thorough engineering text on calculus and analytical geometry. This text is an accumulation of notes on calculus used in the U.S. Army PRE-TPS course in preparation for the US Navy Test Pilot School and also in the PRE-TPS courses for the National Test Pilot School.

2.2 Calculus

2.2.1 Introduction

Calculus is the mathematics of change and motion and is sometimes called the calculus of variations. Calculus provides methods for solving two large classes of problems, one of which involves determining the rate at which a variable quantity is changing. For example, when an inert bomb is dropped from an aircraft, the position of the bomb relative to the aircraft changes with time, therefore, the instantaneous velocity of the bomb also changes with time. Differential calculus is the branch of calculus which treats such problems. Integral calculus on the other hand deals with determining a function when its rate of change is given, which is essentially the reverse of differential calculus. For example in the bomb problem mentioned above, if the instantaneous velocities are given then by using integral calculus, the distance from the aircraft can be determined at any instant of time after release.

2.2.2 Cartesian Co-ordinates

Cartesian Co-ordinates are named after the mathematician Descartes who used this system to locate a point in a two-dimensional area with respect to two reference lines at ninety degrees to each other. The horizontal reference line in Figure 1.1 is called the abscissa or the x axis and the vertical axis normal to the abscissa is called the y axis or the ordinate. An arbitrary point P can be located by a measure from the origin along the x axis and from the origin along the y axis. In analytical geometry the scale of the y and x axes are the same, however in other applications they can be quite different.

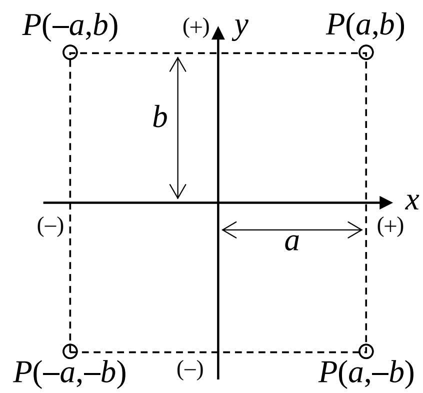


Figure 1.1

Note that the values of x and y can be plus or minus depending on which quadrant the point P is located.

Tutorial

1. Using the Cartesian Co-ordinate system, plot the following points;

   $P_1(3,3)$, $P_2(3, -3)$, $P_3(-3, 3)$, $P_4(-3, -3)$

2. Determine the straight line distance between the following points.

   $(P_1\, P_2)$, $(P_1\, P_3)$ and $(P_1\, P_4)$

3. If a line is driven through the two points (2,3) and (1,1) and it cuts the y axis at $(0,a)$ find the value of a. Also find the value of x where the line crosses the x axis.

4. Determine the angle (a) of the line drawn in No. 3 relative to the x axis.

5. From the sketch of the line in No. 3 find tan α, sin α and cosine α

2.2.3 Increments

If a point $P_1(x_1, y_1)$ is changed to $P_2(x_2, y_2)$ by adding small increments to x and y then the increments are called Δx (delta x) and Δy (delta y) respectively, Figure 1.2.

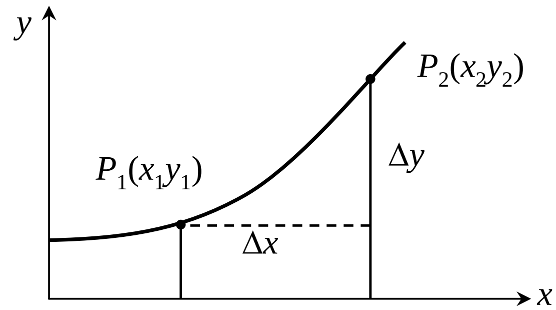


Figure 1. 2 Increments added to $P_1(x_1, y_1)$

Tutorial

1. If a particle starts at position $P_1(-2,3)$ and its co-ordinates receive increments $\Delta x = 5$, $\Delta y = -6$ what will be the new position?

2. Find the starting position of a particle that finishes at position $P_1(a,b)$ after its co-ordinates have received increments $\Delta x = q$ and $\Delta y = r$

3. If a particle moves from point $P_1(-2,5)$ to the y axis in such a way that $\Delta y = 3\Delta x$, what are its new coordinates?

4. A particle moves along the parabola $y = x^2$ from the point $P_1(1,1)$ to the point $P_2(x,y)$. Show that

$$\frac{\Delta y}{\Delta x} = x + 1 \quad \text{if} \quad \Delta x \neq 0$$

2.2.4 Slope of a Straight Line

A straight line is shown in Figure. 1.3 relative to the Cartesian coordinate system and is defined as the straight-line drawn between the Points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. The slope of the line $= m = \frac{\Delta x}{\Delta y}$ $= \frac{y_2 - y_1}{x_2 - x_1}$. Note that the same slope could be obtained by any two points on the line rather than using the points P_1 and P_2. If the angle of the line $P_1\,P_2$ to the horizontal axis x is ϕ then the slope of the line m $= \tan\phi = \frac{\Delta y}{\Delta x}$. The straight line shown in Figure 1.3 has a positive slope i.e. Δy and Δx are positive, a line parallel to the horizontal axis has a slope of zero and a line that slopes downward to the right, Figure 1.4 has a negative slope since Δy is negative for a positive Δx.

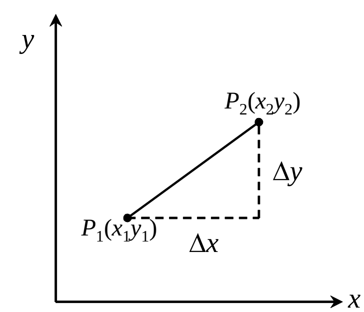


Figure 1.3 Straight Line

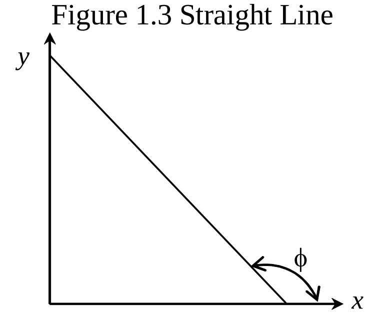


Figure 1.4 Line with Negative Slope

Two parallel lines have equal slopes, as shown in Figure 1.5.

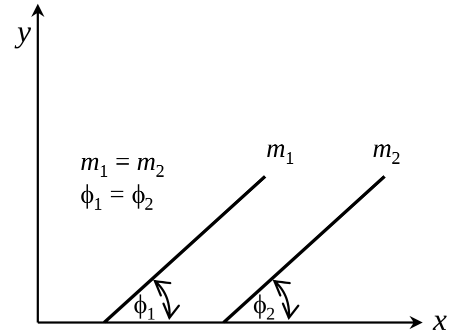


Figure 1.5 Parallel Line

Straight lines perpendicular to each other are shown in Figure 1.6 then

$$m_2 = \tan\phi_2 = \tan\left(\phi_1 + 90^\circ\right) = -\cot\phi_2 = -\frac{1}{\tan\phi} = -\frac{1}{m_1}$$

therefore $$m_2 = -\frac{1}{m_1}$$

for perpendicular straight lines or $m_2 m_1 = -1$

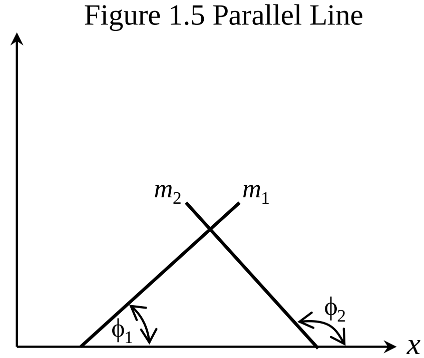


Figure 1.6 Perpendicular Straight Line

Tutorial

1. Find the slope of the line through the points
   a. (3,5), (2,–3)
   b. (–1,2) (4,–3)
   c. (–2,4) (–5,-5)

2. Find the co-ordinates of a point $P_1(x,y)$ which is so located that the line L_1 through; the origin and P_1 has a slope of +2, and the line L_2, through; the point P_2 (–1,0) and P_1 has a slope of +1.

3. Plot the given points and determine analytically whether or not they all lie on a straight line.
   a. $P_1(1,0)$, $P_2(0,1)$, $P_3(2,–l)$
   b. $P_1(–2,–1)$, $P_2(–1,1)$, $P_3(1,5)$, $P_4(2,7)$

4. Given $P_1(0,–l)$, $P_2(4,0)$ and $P_3(3,4)$ show that $P_1\,P_2\,P_3$ is a right triangle.

2.2.5 Equations of a Straight line

A straight line is shown in Figure 1.7 and the slope of the line is defined by

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad y_2 - y_1 = m_1(x_2 - x_1)$$

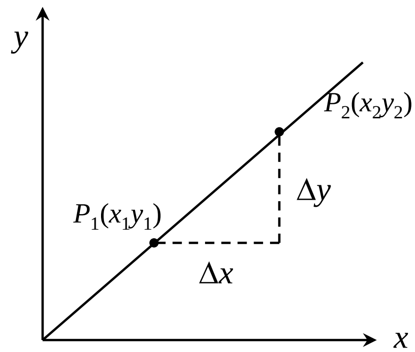


Figure 1.7 Straight Line through the Origin

Since the straight line goes through the origin then $y = mx$. Since at $y = 0$, $x = 0$. Figure 1.8 shows the straight line which does not go through zero. Therefore at $x = 0$, $y = 6$ and the slope is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = m$$

then $y = mx + b$.

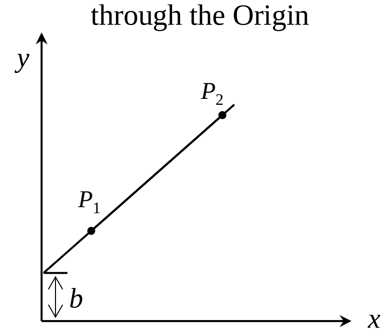


Figure 1.8 Straight Line

Tutorial

1. Find the slope of the straight lines:
   a. $y = 3x + 5$
   b. $x + y = 2$
   c. $3x + 4y = 12$
2. Find the line that passes through the points (1,2) and is parallel to the line $x + 2y = 3$.
3. Find the equation of the line through P, (1,4) and having a slope of 60 degrees.
4. If A, B, C and c' are constants show that
   a. The lines $Ax + By + C = 0$
      $Ax + By + c' = 0$ are parallel
   b. The lines $Ax + By + C = 0$
      $Bx - Ay + c' = 0$ are perpendicular.

5. Let C and F denote, respectively, centigrade and Fahrenheit temperature readings. Given that F ~ C curve is a straight line, find its equation given $C = 0^\circ$, $F = 32^\circ$ and $C = 100^\circ$, $F = 212^\circ$. Also find the temperature at when centigrade equals Fahrenheit.

2.2.6 Functions and Graphs

In the straight line shown in Figure 1.8 and defined by the equation $y = mx + b$, it is obvious that the value of y is a function of x. y is the dependent variable and x is the independent variable and the relationship between y and x is given by the straight line as shown. In many aircraft examples the dynamic response characteristics are often sinusoidal and are similar to the characteristics of the equation $y = \sin x$. Which is given in Figure 1.9.

Note that $y = 0$ when $x = \pm \pi, , \pm 2\pi$ etc

$y = 1$ when $x = \frac{\pi}{2}, \frac{5}{2}\pi$ etc

$y = -1$ when $x = -\frac{\pi}{2}, \frac{3}{2}\pi$ etc

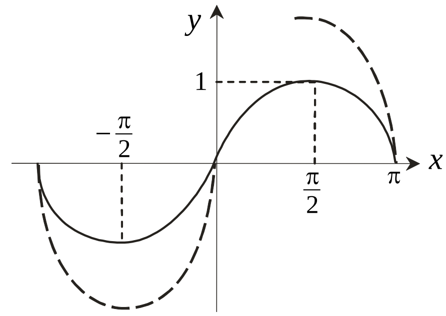


Figure 1.9 Graph of $y = \sin x$

The amplitude of the sinusoidal oscillation can be changed in the equation by changing the value of the constant e.g. $y = 2 \sin x$ is shown by the dotted line in Figure 1.9 which shows that the amplitude is +2 to –2.

Other curves that need to be recognizable are given below. One item that is often ignored are the values of y at negative values Of x, also the equation should also be tested for symmetry, for example if x is replaced by $-x$ and $F(x,y) = F(-x,y)$ then the curve is symmetric with respect to the y axis, Figure 1.10

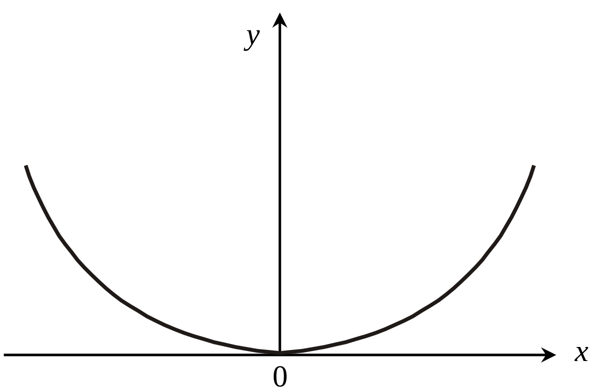


Figure 1.10 Curve Symmetric about the y Axis

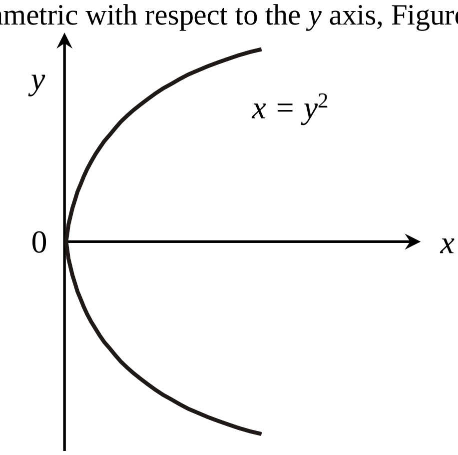


Figure 1.11 Curve Symmetric about the x Axis

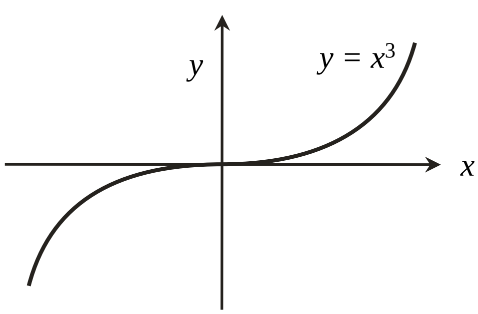


Figure 1.12 $y = x^3$

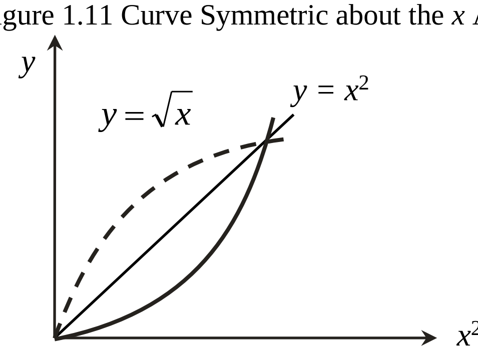


Figure 1.13 Curves of $y = x$, $y = \sqrt{x}$, $y = x^2$

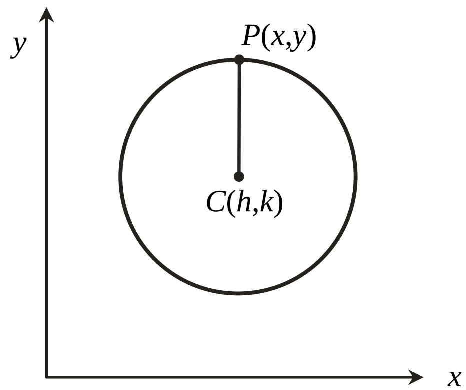


Figure 1.14 The Circle

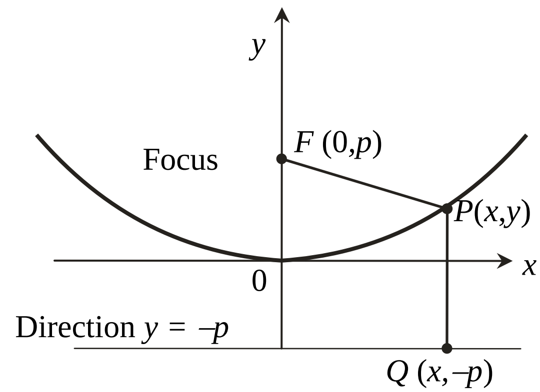


Figure 1.15 Parabola $x^2 = 4py$

A circle is defined as the locus of the points on a plane that are a given distance from a given point.

$$\therefore\ CP = r = \sqrt{(x-h)^2 + (y-k)^2}$$

or $$r^2 = (x-h)^2 + (y-k)^2$$

if the center of the circle is located at the origin then $h = 0$, $k = 0$ and $r^2 = x^2 + y^2$

A parabola is the locus of points in a plane equal distant from a point and a given line. The given point is called the focus of the parabola and the given line is called the directrix.

$$\therefore\ PF = PQ$$

$$\therefore \sqrt{x^2 + (y-p)^2} = \sqrt{(y+p)^2}$$

$$\therefore x^2 = 4py$$

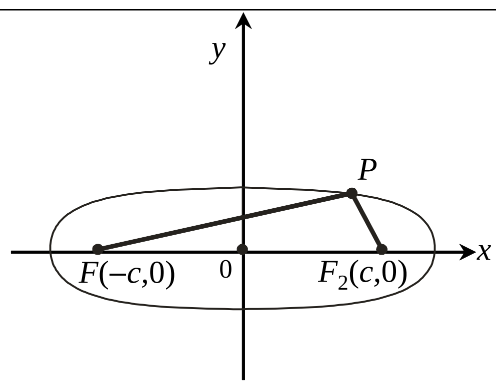


Figure 1.16 Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

An ellipse is the locus of points $P(xy)$ The sum of whose distances from two fixed points is constant. The two fixed points (foci) are taken as $F_1(-c,0)$ and $F_2(c,0)$, and the sum of the distances $PF_1 + PF_2$ is donated by $2a$. Then the coordinates of P must satisfy the equation:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

or $$a - \frac{c}{a}x = \sqrt{(x-c)^2 + y^2}$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

Let $$b = \sqrt{a^2 - c^2}$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

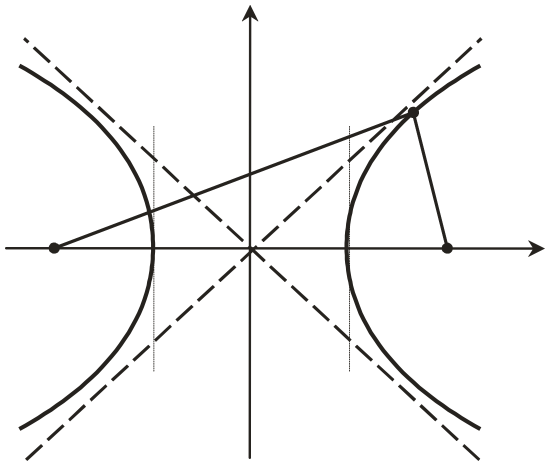


Figure 1.17 Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The hyperbola is the locus of $P(xy)$ if the difference from two points is constant. Taking the fixed points as $F_1(-c,0)$ and $F_2(c,0)$ and the constant equal to $2a$.

Then $\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

i.e. similar to the ellipse but now $a^2 - c^2$ is negative because the difference in two sides of the triangle $F_1 F_2 P$ is less than the third side

i.e. $$2a < 2c$$

$\therefore$ $c^2 - a^2$ is positive and has a positive real square root called b.

$$\therefore b = \sqrt{c^2 - a^2}$$

$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the equation of the hyperbola.

Tutorial

1. Find the center and radius of the given circle

   a. $x^2 + y^2 - 2y = 3$

   b. $x^2 + y^2 + 2x = 8$

   c. $x^2 + y^2 + 2x - 4y + 5 = 0$

2. If V is the vertex and F the focus of a parabola, find the equation of the parabola.

   a. $V(0,0)$, $F(0,2)$

   b. $V(-2,3)$, $F(-2,4)$

   c. $V(1,-3)$, $F(1,0)$

3. Suppose that a and b are positive numbers, sketch the parabolas

$$y^2 = 4a^2 - 4ax$$

$$y^2 = 4b^2 + 4bx$$

4. Sketch the following ellipses

   a. $9x^2 + 4y^2 = 36$

   b. $\dfrac{(x-1)^2}{16} + \dfrac{(y-2)^2}{4} = 1$

5. Show that the equation $\dfrac{x^2}{9-c} + \dfrac{y^2}{5-c} = 1$

   a. represents an ellipse if c is a constant less than 5.

   b. is hyperbola if c is a constant greater than 5 but less than 9.

2.2.7 Slope of a curve

In Figure 1.18 the slope of the line joining the points P_1 and P_2 on the curve shown is

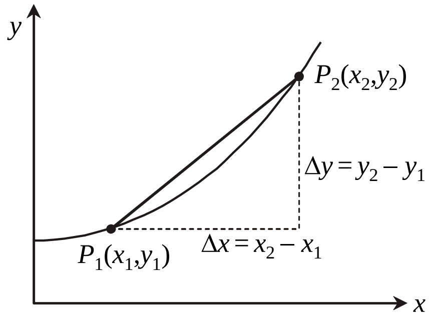


Figure 1.18 Slope of a Curve.

Now, if P_1 is held fixed and the point P_2 is moved closer to P_1, the slope will vary, however as x_2 approaches x_1, or Δx approaches zero the slope will vary less and less and will approach a constant limiting value.

If we assume that the curve in Figure 1.18 is defined by the equation $y = x^3 - 3x + 3$ then points P_1 and P_2 must satisfy the equation

i.e.
$$y_1 = x_1^3 - 3x_1 + 3$$
$$y_2 = x_2^3 - 3x_2 + 3$$

and $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$

then $x_2 = x_1 + \Delta x$ and $y_2 = y_1 + \Delta y$

must also satisfy the equation, therefore

$$(y_1 + \Delta y) = (x_1 + \Delta x)^3 - 3(x_1 + \Delta x) + 3 = x_1^3 + 3x_1^2 \Delta x + 3x_1(\Delta x)^2 + (\Delta x)^3 - 3x_1 - 3\Delta x + 3$$

but $y_1 = x_1^3 - 3x_1 + 3$

$$y_2 = x_2^3 - 3x_2 + 3$$

and subtracting these two equations gives

$$\Delta y = 3x_1^2 \Delta x + 3x_1(\Delta x)^2 + (\Delta x)^3 - 3\Delta x$$

The slope of the curve is $\frac{\Delta y}{\Delta x}$

$$\therefore \frac{\Delta y}{\Delta x} = 3x_1^2 + 3x_1(\Delta x) + (\Delta x)^2 - 3$$

However, if P_2 approaches P_1 then Δx approaches zero i.e. $\Delta x \rightarrow 0$

then $\therefore \left(\frac{\Delta y}{\Delta x}\right)_{\Delta x \rightarrow 0} = 3x_1^2 - 3 = \text{slope} = m$

This limit is the slope of the tangent to the curve, or the slope of the curve at the point (x_1 y_1) since point (x_1 y_1) could be any point on the curve.

Tutorial

1. Find the slope of the following curves at a point (x,y) using the methods of ?????? **Chapter 1.7.** Use the equation of the curve and the equation of the slope to assist in sketching the curve.

a. $y = x^2 - 2x - 3$

b. $y = x^2 - 4x$

c. $y = 2x^3 + 3x^2 - 12x + 7$

d. $y = x^2 (4x + 3) + 1$

e. $y = x^3 - 3x^2 + 4$

2.2.8 Derivative of a Function

The general method of finding the slope of a curve of the general function $y = f(x)$ is derived from the slope determination used in Section 1.7.

Since $y = f(x)$ and $\Delta y = f(x_1 + \Delta x) - f(x)$

then $y + \Delta y = f(x_1 + \Delta x)$

Then by subtracting $y_1 = f(x)$, Δy can be obtained

$$\Delta y = f(x_1 + \Delta x) - f(x)$$

then the slope is $\frac{\Delta y}{\Delta x}$

therefore $\frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

slope $\frac{\Delta y}{\Delta x}$ as Δx approaches zero i.e. $\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x \to 0}$

Therefore $f'(x) = \underset{\Delta x \to 0}{Limit} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

Tutorial

1. Find the derivative of the following functions;

   a. $f(x) = x^2$

   b. $f(x) = x^3$

   c. $f(x) = \frac{1}{2x+1}$

   d. $f(x) = x - \frac{1}{x}$

   e. $f(x) = \frac{1}{\sqrt{x}}$

2.2.9 Limit of a Function

A function $f(x)$ is said to have a limit Δ as $x \to a$ if, as x approaches its limit in any manner whatsoever without assuming the value a, the numerical value of $f(x) - A$ eventually becomes and remains less than any preassigned positive number e, however small.

The following relations hold for limits.

Suppose u, v and w are functions of a variable x, and suppose that

$$\lim_{x \to a} u = A, \ \lim_{x \to a} v = B, \ \lim_{x \to a} w = C$$

Then

a. $\lim_{x \to a} (u + v + w) = A + B + C$

b. $\lim_{x \to a} (u, v, w) = ABC$

c. $\lim_{x \to a} \frac{u}{v} = \frac{A}{B}$ provided B is not zero

example:

1. $\lim_{x \to 3} \frac{x-1}{x^2-x-5} = \lim \frac{3-1}{9-3-5} = 2$

2. $\lim_{x \to A} 4x^2 = 4A^4$

3. $\lim_{x \to 0} \frac{4x^2+3x+6}{2x+1} = \frac{6}{1} = 6$

4. $\lim_{x \to \infty} \frac{6x^2+2x+6}{5x^2-3x-4} = \lim_{x \to \infty} \frac{6+\frac{2}{x}+\frac{1}{x^2}}{6-\frac{3}{x}-\frac{4}{x^2}} = \frac{6}{5}$

Hints:

1. Look for common factor.
2. Substitute the limits.
3. To determine the limit when $x \to \infty$, divide by the highest power of x.

Tutorial

1. If $f(x) = 4 - 2x^2 + x^4$, find $f(0)$, $f(1)$, $f(-1)$, $f(2)$, $f(-2)$
2. Given $f(y) = y^2 - 2y + 6$, show that $f(y + h)$

$$y^2 - 2y + 6 + 2(y - 1)h + h^2$$

Prove each of the following statements;

3. $\lim_{x \to \infty} \frac{4x+5}{2x+3} = 2$

4. $\lim_{h \to \infty} \frac{3h+2xh^2+x^2h^3}{4-3xh-2x^3h^3} = -\frac{1}{2x}$

5. $\lim_{s \to a} \frac{s^4-a^4}{s^2-a^2} = 2a^2$

6. Given $f(x) = ax^2 + bx + c$ show that

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = 2ax+b$$

2.3 Derivatives

2.3.1 Derivatives

A derivative is the rate of change of one variable with respect to another. If the two variables have a linear relationship as shown in Figure 2.1 then the derivative is the slope of the line. **From Chapter 1.8** the slope of the curve is

$$\frac{\Delta y}{\Delta x} \text{ and } \frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

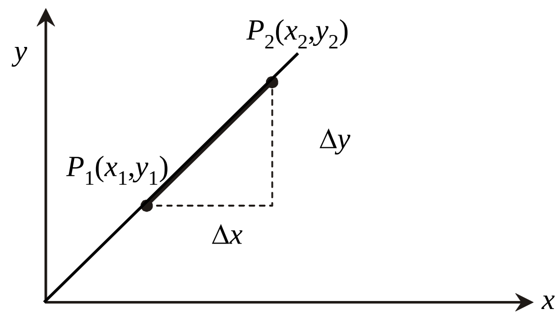


Figure 2.1 Linear Relationships

However, if P_2 moves closer to P_1 then $\Delta x \to 0$ and the slope of the curve

$$\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x \to 0} = \frac{dy}{dx}$$

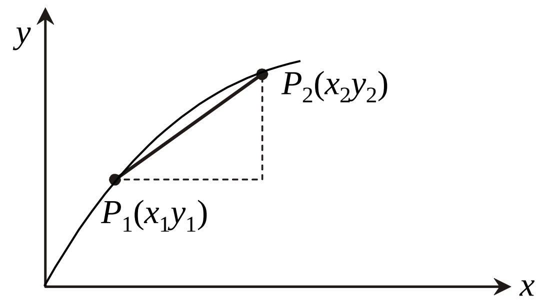


Figure 2.2 Non-Linear Relationships

or the instantaneous slope of the curve at x_1,. If the functional relationship between x and y is not linear as shown in Figure 2.2 then the slope of the line joining P_1 and P_2 will change as x gets smaller and smaller until the slope reaches a constant value when $x \to 0$. The constant value of the slopes $x \to 0$ is the slope of the tangent to the curve at P_1 In aeronautics, the term derivative is used in the context of aircraft stability in that the stability derivative C_{M_α} for example describes the rate of change of the aircraft pitching moment coefficient (C_M) with respect to changes in aircraft angle of attack (α). If the relationship between C_M and α is linear then $\frac{dC_M}{d\alpha}$, which can be written in the shorthand notations C_{M_α}, is the slope of the $C_M \sim \alpha$ curve and is a constant. If the relationship between C_M and α is not linear then C_{M_α} is not constant and values of C_{M_α} are the slopes of the tangents to the $C_M \sim \alpha$ curve at specified values of α. Another classic case of a derivative is the relationship between the aircraft lift coefficient (C_L) and the aircraft angle of attack, as shown in Figure 2.3. The slope of $C_L \sim \alpha$ curve is constant for a large range of α's and is $\frac{dC_L}{d\alpha} = C_{L_\alpha}$ which is called the lift curve slope. For most aeronautical applications $C_L \sim \alpha$ is considered linear and C_L and α can be interchanged in aircraft stability derivatives. For example in Figure 2.3 if α is measured from $\alpha = 0$ then

$$C_L = C_{L_\alpha} \cdot (\alpha + \alpha_{OL})$$

or

$$\alpha = \frac{1}{C_{L_\alpha}} \cdot C_L - \alpha_{OL}$$

Or if α is measured from the zero lift angle then

$$C_L = C_{L_\alpha} \cdot \alpha \quad \text{or} \quad \alpha = \frac{1}{C_{L_\alpha}} \cdot C_L$$

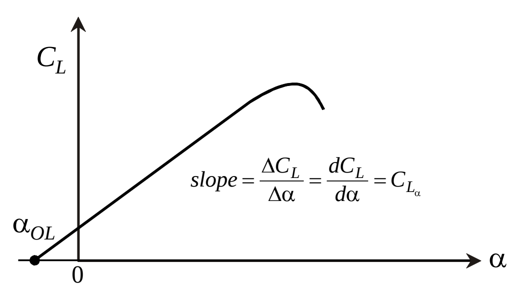


Figure 2.3 Aircraft $C_L \sim \alpha$ Characteristics

Then if the aircraft stability derivative C_{M_α} is considered which is $\frac{dC_M}{d\alpha}$

then $$\frac{dC_M}{d\alpha} = \frac{dC_M}{dC_L} \cdot \frac{dC_L}{dC_L} = \frac{dC_M}{dC_L} \cdot \frac{1}{C_{L_\alpha}}$$

and since C_{L_α} is constant then $C_{M_{LL}}$ is a linear function of C_{M_α}

2.3.2 Derivatives of Algebraic Functions

Since a derivative is an instantaneous rate of change of one variable with respect to another and has been defined as

$$\frac{dy}{dx} = y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Then some general rules for the differentiating of algebraic functions can be determined.

1 The derivation of a constant is zero

i.e. $$\frac{d}{dx}(c) = 0$$

2 The derivative of a variable with respect to itself is unity.

$$\frac{d}{dx}(x) = 1$$

3 The derivative of the algebraic sums of n functions is equal to the same algebraic sum of their derivatives.

$$\frac{d}{dx}(u + v + w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$$

4 The derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function.

$$\frac{d}{dx}(cv) = c\frac{dv}{dx}$$

5 The derivative of the product of two functions is equal to the first function times the derivative of the second, plus the second function time the derivative of the first.

6 The derivative of the product of *n* functions, is equal to the sum of *n* products that can be formed by multiplying the derivative of each function by all the other functions.

$$\frac{d}{dx}(u_1u_2...u_n) = u_2u_3...u_n\frac{du_1}{dx} + u_1u_3...u_n\frac{du_2}{dx} + u_1u_2u_4...u_n\frac{du_3}{dx} + u_1u_2...u_n\frac{du_n}{dx}$$

7 The derivative of a function with a constant exponent is equal to the product of the exponent, the function with the exponent diminished by unity and the derivative of the function.

$$\frac{d}{dx}(v^n) = nu^{n-1}\frac{dv}{dx}$$

8 The derivative of a quotient is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

9 The derivative of an inverse function is equal to the reciprocal of the derivative of the direct function.

$$\frac{dv}{dx}=\frac{1}{\frac{dx}{dy}}$$

10 The derivative of a function of a function is the product of the first function with respect to the third function and the derivative of the third function with respect to the first function.

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}$$

The derivative of a function of a function is often called the chain rule which can be generalized by

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dw}\cdot\frac{dw}{dx}$$

Tutorial

Prove each of the following differentiations:

1. $\frac{d}{dx}\left(3x^4-2x^2+8\right)=12x^3-4x$
2. $\frac{d}{dx}\left(\frac{2}{x}-\frac{3}{x^2}\right)=-\frac{2}{x^2}+\frac{6}{x^3}$
3. $s=\frac{a+bt+ct^2}{\sqrt{t}}\ \frac{ds}{dt}=-\frac{a}{2t\sqrt{t}}+\frac{b}{2\sqrt{t}}+\frac{3c\sqrt{t}}{2}$
4. $s=t\sqrt{a^2+t^2}\ \frac{ds}{dt}=\frac{a^2+2t^2}{\sqrt{a^2+t^2}}$

2.3.3 Implicit Differentiation

When a relationship between *x* and *y* is given by means of an equation not solved for *y*, then *y* is called an implicit function of *x*. For example.

$$x^2 - xy + y^2 = 5$$

by differentiating with respect to *x* gives

$$\frac{d}{dx}\left(x^2\right)-\frac{d}{dx}(xy)+\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(5)$$

$$2x-\left[x\frac{dy}{dx}+y\frac{dx}{dx}\right]+2y\frac{dy}{dx}=0$$

$$\therefore 2x-x\frac{dy}{dx}-y+2y\frac{dy}{dx}=0$$

or

$$\frac{dy}{dx}(2y-x)=y-2x$$

$$\therefore \frac{dy}{dx}=\frac{y-2x}{2y-x}$$

Tutorial

Find $\frac{dy}{dx}$ for each of the following functions:

1. $y = u\sqrt{a^2 - u^2}$, $u = \sqrt{1 - x^2}$, $\frac{dy}{dx} = \frac{x(2u^2 - a^2)}{\sqrt{(a^2 - u^2)(1 - x^2)}}$
2. $y^2 = 2px$
3. $x + 2\sqrt{xy} + y = a$
4. Find the slope of the curve at (2,3)

   $x^2 + 2xy - 3y^2 + 11 = 0$ Answer $= \frac{5}{7}$

2.3.4 Successive Differentiation

The derivative of the first derivative of a function is called the second derivative and is represented by the symbols $\frac{d^2y}{dx^2}$, $f^{\text{II}}(x)$, y^{II}. The derivative of the second derivative is called the third derivative and so on. $\frac{d^3y}{dx^3}$ $f^{\text{III}}(x)$, y^{III} and so on.

Examples

a. Find the sixth and ninth derivative of the function;

$$y = x^7 + 2x^4 - 5x + 1$$

$$y^{\text{I}} = \frac{dy}{dx} = 7x^6 + 8x^3 - 5$$

$$y^{\text{II}} = \frac{d^2y}{dx^2} = 42x^5 + 24x^2$$

$$y^{\text{III}} = \frac{d^3y}{dx^3} = 210x^4 + 48x$$

$$y^{\text{IV}} = \frac{d^4y}{dx^4} = 840x^3 + 48$$

$$y^{\text{V}} = \frac{d^5y}{dx^5} = 2520x^2$$

$$y^{\text{VI}} = \frac{d^6y}{dx^6} = 5040x$$

$$y^{\text{VII}} = \frac{d^7y}{dx^7} = 5040$$

$$y^{\text{VIII}} = \frac{d^8y}{dx^8} = 0$$

$$y^{\text{IX}} = \frac{d^9y}{dx^9} = \text{has no significance}$$

b. For the general case of motion, the velocity is defined at any instant as the rate of change of distance with respect to time i.e.

$$\text{Velocity} = V = \frac{ds}{dt}$$

Also, the acceleration is defined as the rate of change of velocity with respect to time i.e.

$$\text{Acceleration} = a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

for example, it has been found that a body falling freely from rest in a vacuum near the earths surface follows approximately the law $s = 16.1t^2$ where s = height in feet and t = time in seconds.

Therefore, the velocity =

$$\frac{ds}{dt} = \frac{d}{dt}(16.1t^2) = 32.2t\,\frac{ft}{\text{sec}}$$

and the acceleration =

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = 32.2\,\frac{ft}{\text{sec}}$$

Tutorial

Prove each of the following differentiations:

1. $y = 3x^4 - 2x^3 + 6x \qquad \dfrac{d^2y}{dx^2} = 36x^2 - 12x$

2. $u = \sqrt{a^2 + v^2} \qquad \dfrac{d^2u}{dv^2} = \dfrac{a^2}{\left(a^2 + v^2\right)^{\frac{3}{2}}}$

3. $y^2 = 4ax \qquad \dfrac{d^2y}{dv^2} = -\dfrac{4a^2}{y^3}$

4. Given the following equation of rectilinear motion: find the position, velocity and acceleration at the instant indicated.

$$s = 120\,t - 16t^2,\ t = 4 \quad s = 224,\ v = -8,\ a = -32$$

2.3.5 Maximum and Minimum Values of a Function

2.4.1. A function $y = f(x)$ has a relative maximum value for $x = x$ if $f(x\ )$ is greater than any immediate preceding and succeeding values of the function. A function $y = f(x)$ has a relative minimum value for $x = x_0$ if $f(x_0)$ is less than any immediate preceding and succeeding values of the function.

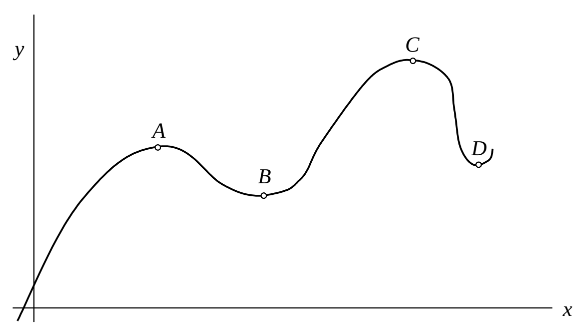
2.4.2. Test for maximum and minimum values of $y = f(x)$

a. First derivative method
   - (1) Find $f'(x)$
   - (2) $f(x)$, is a maximum if $f'(x\ ) = 0$ and $f'(x)$ changes sign from + to – as x increases through x_0.
   - (3) $f(x)$ is a minimum if $f'(x\ ) = 0$ and $f\ (x)$ changes sign from – to + as x increases through x_0.
   - (4) $f(x)$ has neither a maximum or minimum if $f'(x)$ does not change sign.

b. Second derivative method:
   - (1) Find f'(x)
   - (2) $f(x)$ is a maximum if $f'(x) = 0$ and $f'(x)$ = a negative number.
   - (3) $f(x)$ is a minimum if $f'(x) = 0$ and $f'(x)$ = a positive number.

A graphical representation of a curve where the equation is given in rectangular coordinates can illustrate the use of a maximum and minimum values.

It can be seen that at the maximum points (*A* and *C*) the derivative, or slope, is equal to zero $\left(\frac{dy}{dx} = 0\right)$ and progresses form (+) to (–). At the minimum points (*B* and *D*) the derivative is equal to zero and progresses from (–) to (+).



Example:

Find the maximum and minimum points on the curve;

$$y = x^3 - x^2 - 8x + 6$$

Solution.

$$\frac{dy}{dx} = 3x^2 - 2x - 8$$

Setting $\frac{dy}{dx} = 0$ and solving for x gives

Method I.

To determine whether $x = 2$ gives a maximum or a minimum write $\frac{dy}{dx}$ in the factored form.

$$\frac{dy}{dx} = (3x + 4)(x - 2)$$

The first factor is obviously + for all values of x near 2. However,

If x is slightly less than 2, $(x - 2)$ is negative and $\frac{dy}{dx}$ is negative.

If x is slightly more than 2, $(x - 2)$ is positive and $\frac{dy}{dx}$ is positive.

Since the value of $\frac{dy}{dx}$ is zero at the point (2,–6) and changes sign from – to + progressing from less than two to greater than two the point is a minimum. Similarly it may be shown that $\left(-\frac{4}{3}, 12\frac{14}{27}\right)$ is a maximum point.

Method II.

Using the second derivative, $\frac{d^2y}{dx^2} = 6x - 2$

and substituting $x = 2$ and $-\frac{4}{3}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 10 \text{ or positive indicating a minimum point.}$$

Likewise

$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{4}{3}} = -10 \text{ and therefore a maximum exists at } x = -\frac{4}{3}$$

Direction of Bending:

a. If the arc of a curve, $y = f(x)$ is concave upward, then:
   (1) At each of its points the arc lies above the tangent.
   (2) y^{I} is increasing (becoming more positive) as x increases along the curve.
   (3) y^{II} is positive at all points on the curve.

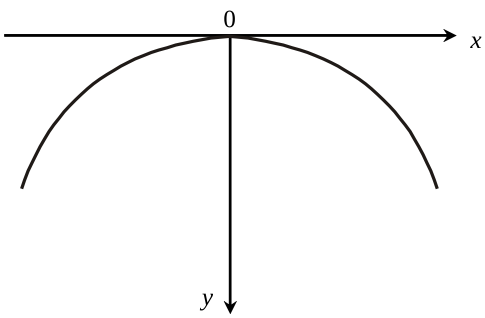
then;

b. If the arc of a curve, $y = f(x)$ is concave downward,

(1) At each of its points the arc lies below the tangent.

(2) y^{I} is decreasing (becoming more negative) as x increases along the curve.

(3) y^{II} is negative at all points on the curve.



$$y = -x^2$$
$$\frac{dy}{dx} = -2x$$
$$\frac{d^2y}{dx^2} = -2$$

Figure 2.5

Thus it is seen that y^{II} determines the direction of bending of a curve and therefore can be utilized to determine extreme values.

(4). Point of Inflection

(a). The point of inflection is the point on a curve at which the curve is changing from concave downward to concave upward or vice versa.

(b). Test for point of inflection

1. $f^{II}(x) = 0$ or becomes infinite
2. $f^{I}(x)$ changes sign as x increases through the point.

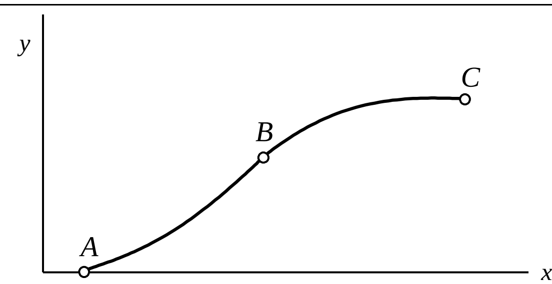


Figure 2.6

The point of inflection at *B* is where the curve goes from concave upward to concave downward.

Tutorial

Examine for maximum and minimum values for x and y.

1. $y = x^3 + 2x^2 - 15x - 20$
2. $y = 3x - 2x^2 - \frac{4}{3}x^3$

   $x = \frac{1}{2}$ gives $y = \frac{5}{6}$ Maximum Point

   $x = -\frac{3}{2}$ gives $y = -\frac{9}{2}$ Minimum Point

3. A rectangular garden is to be laid out along a neighbor's lot and is to contain 432 sq rd. If the neighbor pays for half the dividing fence, what should be the dimensions of the garden so the cost to the owner of enclosing it may be a minimum? Ans. 18 rd x 24 rd.

2.3.6 Differentials

The differential of a function equals its derivative multiplied by *the differential of the independent variable* as previously explained.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

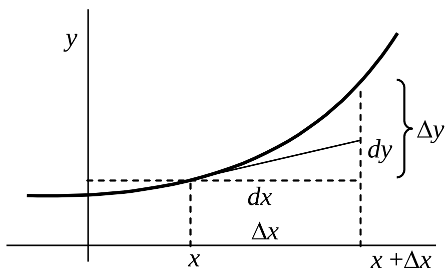
where $$dy = \lim_{\Delta y \to 0} \Delta y$$

and $$dx = \lim_{\Delta x \to 0} \Delta x$$

then $$f'(x)dx = \frac{dy}{dx}(dx)$$

dy = differential of y

Note: Both dy and dx are infinitesimals.



Examples:

1. $y = ax^n$ $\quad \frac{dy}{dx} = nax^{n-1}$ $\quad dy = nax^{n-1}\,dx$
2. $y = 4x^2$ $\quad dy = 8x\ \ dx$

Use of differentials.

1. Approximations. If $dx = x$ is relatively small when compared to x, they dy is a good approximation of Δy.

   Example: Find an appropriate formula for the volume of a thin cylindrical shell with open ends if the radius is r, the length h, and the thickness t.

   a. Exact formula

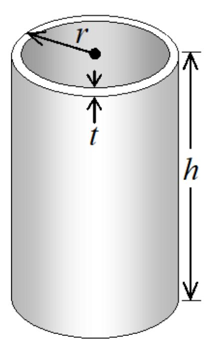
   $V_0 = \pi v r h$

   $V_1 = \pi(r - t)^2\, h$

   $V_1 = \pi\, h\, (r^2 - 2tr + t^2)$.

   $V_s = V_0 - V_1 = 2\,\pi r t h - \pi h t^2$

   $V_s = (2\pi h r t) - \pi h t^2$



Approximate value by use of differentials $V = \pi r^2 h$

$$dV = 2\pi r dr h + \pi r^2 dh$$

where $dr = t$ and $dh = 0$

Therefore $dv = 2\pi rht$

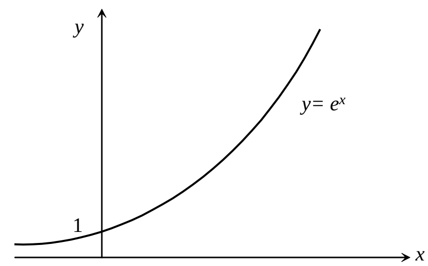
Tutorial

1. A box is to be constructed in the form of a cube to hold 1000 cu ft. How accurately must the inner edge be made so that the volume will be correct to within 3 cu ft? Ans. Error ≤ 0.01 ft.
2. How exactly must the diameter of a circle be measured in order that the area is correct to within 1 per cent? Ans. Error $\leq \frac{1}{2}$%.

2.3.7 Differentiation of Transcendental Functions

The word transcental means to transcend normal experience or of a higher degree.

a. The exponential function $y = e^x$ or $x = \ln y$



where $e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$

$e = 2.71828\ldots$ (irrational)

if $x = e^y$ then $\frac{dx}{dy} = e^y$

b. The natural logarithmic function

$x = \ln y$ is the inverse of $y = e^x$

where $\ln y = \frac{\log y}{\log e}$, and the common logarithmic is

$\ln y \log e = \log y$

Formulas:

The following nine formulas should be committed to memory.

1. $\frac{d}{dx}(\ln v) = \frac{\frac{dv}{dx}}{v} = \frac{1}{v}\frac{dv}{dx}$
2. $\frac{d}{dx}(\log v) = \frac{\log e}{v}\frac{dv}{dx}$
3. $\frac{d}{dx}(a^v) = a^v \ln a \frac{dv}{dx}$
4. $\frac{d}{dx}(e^v) = e^v \frac{dv}{dx}$
5. $\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + u^v \ln u \frac{dv}{dx}$
6. $\frac{d}{dx}(\sin v) = \cos v \frac{dv}{dx}$
7. $\frac{d}{dx}(\cos v) = -\sin v \frac{dv}{dx}$
8. $\frac{d}{dx}(\tan v) = \sec^2 v \frac{dv}{dx}$
9. $\frac{d}{dx}(ctnv) = -\csc^2 v \frac{dv}{dx}$

Tutorial

Differentiate each of the following functions:

1. $y = \ln(ax + b)$ Answer: $\frac{dy}{dx} = \frac{a}{ax+b}$
2. $y = \ln(ax + b)^2$ $\frac{dy}{dx} = \frac{2a}{ax+b}$
3. $y = \ln x^3$ $\frac{dy}{dx} = \frac{3}{x}$
4. $y = \sin a\,x$ $y^{I} = a \cos ax$
5. $s = \tan 3\,t$ $s^{I} = 3 \sec^2 3t$

2.4 Integration

There are two kinds of integration; one which is the reverse process of differentiation and the other is finding the total area bounded by curves or the volume of various solids, etc. The reverse process of differentiation is to find a function whose derivative is given and is called the indefinite integral. The definite integral is a summation or totalizing of areas or volumes, for example, between definite limits.

2.4.1 The Indefinite Integral

If $f(x)$ is a function whose derivative is $f'(x)$ then $f(x)$ is called the integral of $f^{I}(x)$. If $f(x)$ is the integral of $f'(x)$ a constant of integration must be added to $f(x)$ i.e. $f(x) + C$. Since any constants in $f(x)$ would disappear when differentiated to $f'(x)$. The function $f(x)$ which is an integral of the differential $f'(x)$ is indicated by the integral sign $\int$ in front of the differential expression i.e.

$$\int f'(x)\,dx = f(x) + C$$

Example Find y as a function of x when given the differential equation

$$\frac{dy}{dx} = 2x$$

from experience of differentiation and knowing that integration is the reverse process of differentiation then

$$\int \frac{dy}{dx} = \int 2x$$

therefore $y = x^2 + c$

In this text we will restrict the differential equations to be solved to first order equations, higher order equations are addressed in other texts. Integration, unfortunately, primarily involves recognization of the form of the integral and the application of special technique and rules to obtain a solution. The following are some standard forms that must be remembered. The terms u and v denote differential functions of an independent variable (say x) and a, n and c are constants.

1 $\int du = u + c$

2 $\int adu = a\int du = au + c$

3 $\int (u+v)dx = \int u\,dx + \int v\,dx$

4 $\int u^n du = \frac{u^{n+1}}{n+1} + c$

5 $\int(du + dv - dw) = \int du + \int dv - \int dw$

6 $\int\left(\frac{du}{u}\right) = \ln u + c$

7 $\int a^u du = \frac{a^u}{\ln a} + c$

8 $\int e^u du = e^u + c$

9 $\int \sin u\, du = -\cos u + c$

10 $\int \cos u\, du = \sin u + c$

11 $\int \tan u\, du = \ln \sec u + c$

12 $\int \cot u\, du = \ln \sin u + c$

Examples

1. $\int \frac{dx}{x^2} = \int x^{-2} dx = \frac{x^{-2+1}}{-2+1} + C = \frac{x^{-1}}{-1} + C = \frac{-1}{x} C$

2. $\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-\frac{2}{3}} dx = \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + C = \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + C = 3x^{\frac{1}{3}} C$

Tutorials

Verify the following integrations.

1. $\int x^4 dx = \frac{x^5}{5} + C$
2. $\int \frac{dx}{x^2} = -\frac{1}{x} + C$
3. $\int x^{\frac{2}{3}} dx = \frac{3x^{\frac{5}{3}}}{5} + C$
4. $\int \frac{4x^2 dx}{\sqrt{x^3+8}} = \frac{8\sqrt{x^3+8}}{3} + C$
5. $\int \frac{dx}{2+3x} = \frac{\ln(2+3x)}{3} + C$
6. $\int \frac{tdt}{a+bt^2} = \frac{\ln(a+bt^2)}{2b} + C$
7. $\int \frac{e^\theta d\theta}{a+be^\theta} = \frac{\ln(a+be^\theta)}{b} + C$
8. $\int 6e^{3x} dx = 2e^{3x} + C$

9. $\int a^{ny}dy = \dfrac{a^{ny}}{n \ln a} + C$

10. $\int \sqrt{e^t}\,dt = 2\sqrt{e^t} + C$

11. $\int sin\,ax \cdot cos\,ax \cdot dx = \dfrac{sin^2\,ax}{2a} + C$

12. $\int \tan\dfrac{x}{2} \cdot \sec^2\dfrac{x}{2} \cdot dx = \tan^2\dfrac{x}{2} + C$

13. $\int e^{\frac{x}{n}}dx = ne^{\frac{x}{n}} + C$

14. $\int e^{\sin x} \cdot \cos dx = e^{\sin x} + C$

15. $\int a^x e^x dx = \dfrac{a^x e^x}{1 + \ln a} + C$

16. $\int \dfrac{dx}{\sin^2 x} = -\text{ctn}\;x + C$

2.4.2 The definite Integral

The definite integral is used to determine the unique value of a function when summated or integrated between definite limits. Since the value of the integration is unique the constants of integration disappear e.g.

$$\int_a^b f'(x)dx = f(x)]_a^b = f(b) - (a)$$

The above is the integral of f^1 (x) from a bottom limit to an upper limit of *b* and is *a* definite integral.

Example

1. Evaluate $$\int_0^3 x^2 dx = \left.\frac{x^3}{3}\right]_a^b = \frac{1}{3}\left[3^3 - 0^3\right] = 9$$

The definite integral process is very valuable in finding the area bounded by curves. To find the area bounded by a curve $y = x^3$, a. line $x = 4$ and the x axis as shown in Figure 3.1. The definite integral is used.

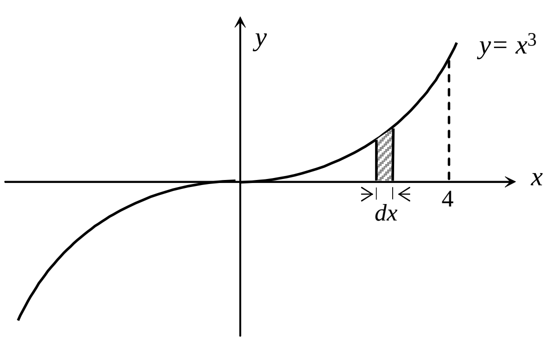


Figure 3.1 Curve of $y = x^3$

To set up the equation the following process is used:-

1. Define a basic incremental area as shown in Figure 3.1

$$\therefore\ dA = y\,dx$$

Note: Although both sides of the incremental area have different values of y as $dx \to 0$ the values of y go to a constant value.

2. Summate or integrate all the incremental areas between the limits of $x = 0$ and $x = 4$.

$\int_0^4 dA$ = area bounded by the curve $y = x^3$, $x = 4$ and the x axis = A.

$$\int_0^4 y dx$$

Since $y = x^3$

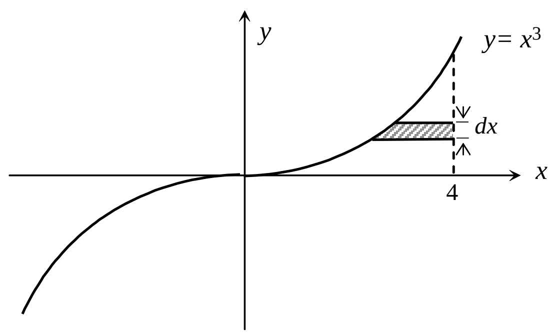
$$\therefore\ A = \int_0^4 x^3 dx = \left.\frac{x^4}{4}\right]_0^4 = 64 - 0 = 64 \text{ square units}$$

The problem can also be solved by taking the incremental area $dA = x\,dy$ as shown in Figure 3.2.

The integral now becomes

$$\int da = \int_0^{64} (4 - x) dy$$

Note: The limits must be changed to cover the range of variations in y. i.e. at $x = 0$, $y = 0$ and at $x = 4$, $y = 64$. Also x must be expressed as a function of y.



Since $y = x^3$ then $x = y^{\frac{1}{3}}$

Therefore, the integral becomes:

$$A = \int_0^{64} \left(4 - y^{\frac{1}{3}}\right) dy$$

$$= 4y - \tfrac{3}{4} y^{\frac{4}{3}} \Big]_0^{64}$$

$$= \left[4(64) - \tfrac{3}{4}(64)^{\frac{4}{3}}\right] - \left[4(0) - \tfrac{3}{4}(0)^{\frac{4}{3}}\right]$$

$$= 256 - \frac{3(256)}{4}$$

$$= 256 - 192$$

$$= 64 \text{ square miles}$$

Obviously taking the incremented area as $x\,dy$ presents more problems than taking the incremented area as $y\,dx$, therefore, before setting up any strip for an integration problem, make things as simple as possible.

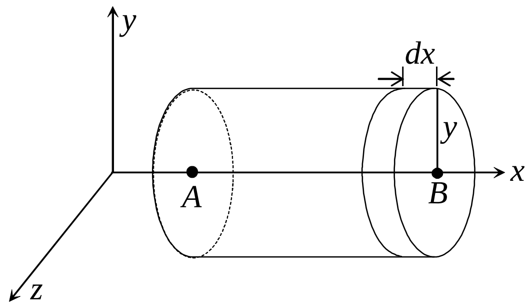


Figure 3.3 Sketch of Solid Cylinder

The definite integral can also be utilized in computing the volume and surface areas of a solid of revolution.

$$\therefore \text{ Volume } = Vx = \pi \int_A^B y^2\,dx$$

$$\text{and Surface Area } = Sx = 2\pi \int_A^B y\,dx$$

2.4.3 Integration by Parts

The fundamental formula for integration by parts is derived by the reverse process of differentiation of the product of two variables,

$$d(vu) = u\,dv + v\,du$$

or

$$u\,dv = d\,(uv) - v\,du$$

and by integrating the reverse formula is obtained

$$\int u dv = uv - \int v\,du + c$$

Example

To find the integral of $\int x \cos x\,dx$

let $u = x$ and $dv = \cos x\,dx$

then $du = dx$ and $v = \sin x$

And by substitution into the original formula

$$\int u dv = uv - \int v\,du + c$$

given

$$\int x \cos x dx = x \sin x - \int \sin x dx$$

$$\int x \cos x dx = x \sin x + x \cos x + C$$

As can be seen from the above example it is very important to pick *u* skillfully such that when *du* is obtained the integral is in a form that is recognizable and to which a known formula can be applied. Sometimes the first integration by parts will not give a solution but each integration will reduce the algebraic portion by one degree, repeat integrating by parts until a solution is obtained.

Tutorials

Work out integrals.

1. $\int x \sin x dx = \sin x - x \cos x + C$

2. $\int x \cos nx dx = \frac{\cos nx}{n^2} + \frac{x \sin nx}{n} + C$

3. $\int x a^x dx = a^x \left[\frac{x}{\ln a} - \frac{1}{\ln^2 a} \right] + C$

4. $\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$

Work out the following integrals.

1. $\int_0^a \left(a^2 x - x^3\right) dx = \frac{a^4}{4}$

2. $\int_0^4 \frac{ax}{\sqrt{9-2x}} dx = \frac{14}{3} a$

Find the area bounded by the given curve, the *x* axis, and the given ordinate.

$$y^2 = 4x + 16,\ x = -2,\ x = 0$$

Find the area included between two parabolas $y = ax$ and $x = by$,

Answer: $\frac{1}{3}ab$

A square is formed by the coordinate axis and parallel lines through the point (1,1). Calculate the ratio of the larger to the smaller of the two areas into which it is divided by the following curve.

$$y = x^2$$

Answer: 2

2.4.4 Integration by Substitution

Since it is known that $\int u^2 du = \frac{u^{n+1}}{n+1} + C$ if $n \neq 1$

and $\int u^2 du = \ln|u + C|$ if $n = -1$

i.e. $\int \frac{1}{u} du = \ln|u + C|$

then by substitution the integral can often be put in the form shown above which can be easily integrated.

Example 1.

Integrate $\int \sin^n ax \cos ax\, dx$

Let $u = \sin ax$

therefore $du = a \cos ax\, dx$

and by substituting $\int \sin^n ax \cos ax\, dx = \frac{1}{\mathrm{a}} \int u^n du$

Note: The $\frac{1}{a}$ is required since $du = a \cos ax\, dx$

$$= \frac{1}{a} \frac{u^{n+1}}{n+1} + C \quad \text{if} \quad n \neq -1$$

and $$= \frac{1}{a} \ln|u| + C \quad \text{if} \quad n = -1$$

therefore if $n \neq -1$ $\int \sin^n ax \cos ax\, dx = \frac{1}{a}\frac{\sin^{n+1} ax}{n+1} + C$

and if $n = -1$ $\int \cot ax\, dx = \frac{1}{a}\ln|\sin ax| + C$

Example 2

Integrate $\int \sin^3 x\, dx$

Substituting $u = \sin x$ will not work in this case since there is no $\cos x$ to go along with the dx to give du however, $\sin^3 x = \sin x \sin^2 x$, and since $\sin^2 x = (1 - \cos^2 x)$

then $\int \sin^3 x\, dx = \int \sin x(1 - \cos^2 x)dx$

and letting $u = \cos x$, $du = -\sin x\, dx$

Substituting gives

$$\int \sin^3 x\, dx = \int (1 - u^2)(-du)$$

$$= \int (u^2 - 1)du$$

$$\int \sin^3 x\, dx = \frac{\cos^3 x}{3} - \cos x + C$$

Tutorials

1. Integrate $\int \sec x \tan x\, dx$ $\left[\frac{1}{\cos x} + C = \sec x + 2\right]$
2. Integrate $\int \tan^4 x\, dx$ $\left[\frac{\tan^3 x}{3} - \tan x + x + C\right]$
3. Integrate $\int \sin^2 3x \cdot \cos 3x\, dx$ $\left[\frac{\sin^3 3x}{9}\right]$
4. Integrate $\int \cos^{\frac{2}{3}} x \sin^5 x\, dx$ $\left[-\cos^{\frac{5}{3}} x\left(\frac{3}{5} - \frac{6}{11}\cos^2 x + \frac{3}{17}\cos^4 x\right) + C\right]$

2.4.5 Integration by Trigometric Substitution

If the integrals involve terms such $\sqrt{a^2 - u^2}, \sqrt{a^2 + u^2}, a^2 \pm u^2$ etc. then by using trigometric substitution and knowing certain trignometric identities solutions can be obtained. The identities that should be remembered are as follows:

$\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta - \cos^2\theta = 1$

$\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

Example

$$\int \frac{du}{\sqrt{a^2 - u^2}}$$

substitute $u = a \sin \theta \quad \therefore\ du = a \cos \theta\, d\theta$

and $\theta = \sin^{-1} \frac{u}{a}$

therefore $a^2 - u^2 = a^2 (1 - \sin^2\theta) = a^2 \cos^2 \theta$

then $$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{a \cos \theta d\theta}{\sqrt{a^2 \cos^2 \theta}} = \int \frac{a \cos \theta d\theta}{\pm a \cos \theta} = \pm \int d\theta = \pm\theta + C$$

$$\therefore \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$$

Tutorials

1. $\int \frac{du}{\sqrt{a^2 + u^2}}$ $\left[\ln \left| \sqrt{a^2 + u^2} + u \right| + C \right]$
2. $\int \frac{du}{\sqrt{u^2 - a^2}}$ $\left[\ln \left| u + \sqrt{u^2 - a^2} \right| + C \right]$

2.4.6 Partial Differentiation

There are many instances in science and engineering where a quantity is defined by more than one other variable rather than the one variable considered so far in this text. An example is the volume of a right circular curve which is

$$V = \frac{1}{3}\pi r^3 h$$

where V is the volume which is defined in terms of the two variables of radius r and height h. To determine the variation of volume V with changes in radius r for cylinders of a constant height since h is held constant the derivative of V with respect to r is called a partial derivative and is represented by $\frac{\partial V}{\partial r}$.

Consider the single value (one value of z for each admissible pair of values of x and y) function $z = f(x,y)$. Suppose x is changed by the amount x keeping y fixed. The change in the value of is

$$\Delta z = f(x + \Delta x,y) - f(x,y)$$

and dividing by x gives $$\frac{\Delta z}{\Delta x} = \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

and $$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

This limit is called the partial derivative of z with respect to x and represents the instantaneous rate of change of z relative to x when y is held constant. Similarly

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta x,) - f(x,y)}{\Delta y}$$

The usual formulae can be used to find these derivatives since when differentiating with respect to x, y is held constant and vice versa. A typical representation of partial derivatives is shown in Figure 4.1 where

$z = f(xy)$ and represents a curved surface. The partial derivative of *z* with respect to *x* at a constant value of *y* as shown would be $\frac{\partial z}{\partial x}$ and represents the local tangent to the curved surface parallel to the *xz* plane at a constant value of *y*. Similarly $\frac{\partial z}{\partial y}$, represents the local slope to the curved surface parallel to the *zx* plane at a constant value of x.

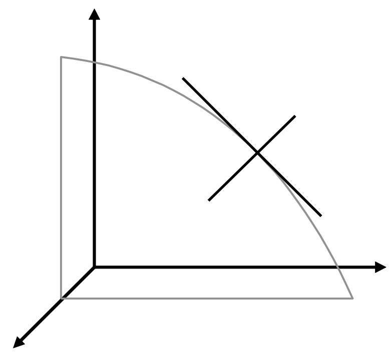


Figure 4.1 $z = f(x,y)$

The partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of a function $z = f(x, y)$ are themselves, in general, functions of *x* and *y* that in turn have partial derivatives with respect to *x* and *y*. The following notation is used to denote these second partial derivatives;

$$\frac{\partial}{\partial x}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y) \qquad \frac{\partial}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2} = f_{xx}(x,y)$$

$$\frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x,y) \qquad \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x,y)$$

This Process can be carried out continually until the function is no longer continuous.

Examples:

1. $z = \frac{x^2}{y} + \frac{y^2}{x}$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = \frac{2x}{y} - \frac{y^2}{x^2}$$

$$\frac{\partial z}{\partial y} = -\frac{x^2}{y^2} + \frac{2y}{x}$$

2. $z = 2x^2 - 3xy + 4y^2$ find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$

$$\frac{\partial z}{\partial x} = 4x - 3y \qquad \frac{\partial^2 z}{\partial x^2} = 4$$

$$\frac{\partial z}{\partial y} = -3x + 8y \qquad \frac{\partial^2 z}{\partial x \partial y} = -3$$

Tutorial

Work out the following partial derivatives.

1. $z = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$

   Answer $\frac{\partial z}{\partial x} = 2Ax + By + D$

   $\frac{\partial z}{\partial y} = Bx + 2Cy + E$

2. $f(x,y) = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$

   Answer $f_x(x,y) = 3(Ax^2 + 2Bxy + Cy^2)$

   $f_y(x,y) = 3(Bx^2 + 2Cxy + Dy^2)$

3. $u = xy + yz + 2x$

   Answer $u_x = y + 2$

   $u_y = x + y$

   $u_z = y$

2.5 Total Differential and Total Derivatives

A function $z = f(x,y)$ that consists of two independent variables z and y has the partial differential of z with respect to x defined as $dz\frac{\partial z}{\partial x}dx$ and the *partial differential* of z with respect to y defined as $dz\frac{\partial z}{\partial y}dy$. The *total differential* is the sum of the partial differentials or

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

If $z = f(x,y)$ is continuous with continuous partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and if x and y are differentiable functions $x = \phi(u)$ and $y = \psi(u)$ of the variable u then $\frac{\partial z}{\partial u}$ is called the *total derivative* of z with respect to u and is

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{dx}{du} + \frac{\partial z}{\partial y}\cdot\frac{dy}{du}$$

Examples:

1. Find the total differential of $z = x^3y + x^2y^2 + xy^3$

$$\frac{\partial z}{\partial x} = 3x^2y + 2xy^2 + y^3$$
$$\frac{\partial z}{\partial y} = x^3 + 2x^2y + 3xy^2$$
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$
$$dz = \left(3x^2y + 2xy^2 + y^3\right)dx + \left(x^3 + 2x^2y + 3xy^2\right)dy$$

2. Find the total derivative $\frac{dz}{dt}$, when $z = x^2 + 3xy + 5y^2$, $x = \sin t$, $y = \cos t$

$$\frac{\partial z}{\partial x} = 2x + 3y, \quad \frac{\partial z}{\partial y} = 3x + 10y$$
$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = -\sin t,$$
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$
$$\frac{dz}{dt} = (2x + 3y)(\cos t) + (3x + 10y)(-\sin t)$$
$$\frac{dz}{dt} = (2x + 3y)(\cos t) - (3x + 10y)\sin t$$

Tutorial

Assuming the characteristic equation of a perfect gas to be $vp = Rt$ where v = volume, p = pressure, t = absolute temperature, and R a constant, what is the relation between the differentials dv, dp, dt ?

Answer: $vdp + pdv = Rdt$

2.5.1 Multiple Integrals

The opposite process to partial differentiation in calculus is partial integration. A partial integration involves integrating the expression considering only one variable and holding the other variables constant. The integration is then continued considering another variable and holding the others constant, this process is continued until all the variables have been integrated. By using this partial integration process double, triple a multiple integrals are obtained.

Example 1 :

If $\frac{\partial z}{\partial x} = 6x + 2y + 5$ then integrating with respect to x gives $z = 3x^2 + 2xy + 5x + \psi$, where ψ denotes the constant of integration. Since y was constant then ψ can involve y and is replaced by $\psi(y)$.

Successive partial integration is involved when

$$\frac{\partial^2 z}{\partial x \partial y} = x^3 + y^4 \text{ for example}$$

To find z then a double integral must be performed as shown

$$z = \iint \left(x^3 + y^4\right) dy dx$$

Double integrals are performed from the-inside and progress outward; for example if y is considered the variable and x is held constant to perform the first integral then x is considered the variable and y is held constant in the second integrate:

$$\frac{\partial z}{\partial y} = x^3 y + \frac{y^5}{5} + \psi(x)$$

Then integrating with respect to x gives

$$z = \frac{x^4}{4} y + xy^5 + \psi(x) + \psi(y)$$

2.6 Definite Multiple Integrals

If limits are placed on each of the variables then the constants of integration are eliminated.

Example

Solve. $$\int_1^2 \int_1^3 \int_2^3 x^2 yz^3 dz\, dy\, dx$$

Since the integration is started from the inside then the expression is integrated with z as the variable with x and y constant, i.e.

$$\int_1^2 \int_1^3 \left[\int_2^3 x^2 yz^3 dz \right] dy dx = \int_1^2 \int_1^3 \left[\frac{x^2 yz^4}{4} \right]_2^3 dy\, dx$$

$$= \int_1^2 \int_1^3 \frac{x^2 y}{4} \left[3^4 - 2^4\right] dy\, dx$$

$$= \frac{65}{4} \int_1^2 \int_1^3 x^2 y\, dy\, dx$$

Then integrating with respect to y with x constant gives

$$= \frac{65}{4} \int_1^2 \left[x^2 \frac{y^2}{2} \right]_1^3 dx$$

$$= \frac{520}{8} \int_1^2 x^2 dx$$

$$= 65 \left[\frac{x^3}{3} \right]_1^2 = \frac{65}{3}(7) = \frac{155}{3}$$

Tutorials

Work out the following definite integrals.

1. $\int_0^1 \int_0^2 (x+2)\,dy\,dx = 5$

2. $\int_0^a \int_0^{\sqrt{x}} dy\,dx = \frac{2}{3}a^{\frac{3}{2}}$

3. $\int_0^{-1} \int_{y+1}^{2y} xy\,dy\,dx = \frac{11}{24}$

Volume 1 – Math & Physics for Flight Testers

Chapter 3

Differential Equations

Table of Contents

3.1 Introduction

This chapter reviews the mathematical tools and techniques required to solve differential equations. Study of these operations is prerequisite for courses in aircraft flying qualities, equations of motion, and linear control systems taught at the National Test Pilot School. Only analysis and solution techniques which have direct application for work at the school will be covered.

The following terms will be used extensively:

Differential Equation: An equation relating two or more variables in terms of derivatives i.e.; Newton's second law can be expressed as [$fdt = mdv$] or $\left[f = m\frac{dv}{dt}\right]$.

Independent Variables: Variables that are not dependent on other variables. In a differential equation, the independent variables are on the right-hand side of the equation and have derivatives taken with respect to them. The independent variable is time for Newton's second law.

Dependent Variables: Variables that are dependent on other variables. In a differential equation, the dependent variables are the variables on the left-hand side of the equation that have their derivatives taken with respect to another variable. The other variable, usually time in our study, is the independent variable. The in Newton's second law the independent variable is velocity.

Ordinary Differential Equation: A differential equation with only one independent variable i.e. Newton's second law or Ohm's law: $V = R\frac{dq}{dt}$.

Partial Differential Equation: A differential equation with more than one independent variable. An example of this is the diffusion equation, $\frac{\partial t}{\partial x} = k\frac{\partial t}{V}$.

Order: An n^{th} derivative is a derivative of order n. A differential equation has the order of its highest derivative. $3\frac{d^3x}{dt^3} + 7\frac{d^2x}{dt^2} - 10\frac{dx}{dt} = 0$ is a 3rd order ordinary differential equation.

Degree: The exponent of a differential term. The degree of a differential equation is the exponent of its highest order derivative. $8\left[\frac{dx}{dt}\right]^4 + 32g = g$ is a 4th degree ordinary differential equation.

Linear Differential Equation: A differential equation in which the dependent variable and all its derivatives are only first degree, and the coefficients are either constants or functions of the independent variable. Examples are $18t\frac{d^2y}{dt} + 6\frac{dy}{dt} + gt = 0$ and $7\frac{d^2y}{dt} + 4.17\frac{dy}{dt} + gt + y = 0$.

Linear System: Any physical system that can be described by a linear differential equation of order n which contains n arbitrary constants.

Solution: Any function without derivatives that satisfies a differential equation.

General Solution: Any function without derivatives which satisfies a differential equation of order n which contains *n* arbitrary constants.

Basic Differential Equation Solution

Unfortunately, there is no general method to solve all types of differential equations. The solution of a differential equation involves finding a mathematical expression (without derivatives) which satisfies the differential equation. It is much easier to determine if a candidate solution to a differential equation is a true solution than to determine a likely candidate. For example, given the linear first order differential equation:

$$\frac{dy}{dx} - x = 4 \tag{3.1}$$

and a possible candidate solution:

$$y = \tfrac{1}{2}x^2 + 4x + C \tag{3.2}$$

It is easy to differentiate equation 3.2 and substitute into equation 3.1 to see if it is a true solution. The derivative of equation 3.2 is:

$$\frac{dy}{dx} = x + 4 \tag{3.3}$$

Substituting equation 3.3 into equation 3.1, we get:

$$(x+4) - x \equiv 4$$

Therefore equation 3.2 is a solution to equation 3.1. It is interesting that, in general, solutions to linear differential equations are not linear functions. Note that equation 3.2 is not an equation of the form $y = mx + b$ which represents a straight line.

There are several methods in use to solve differential equations. The methods to be discussed in this chapter are:

1. *Direct Integration*
2. *Separation of Variables*
3. *Exact Differential Integration*
4. *Integrating Factors*
5. *Operator Techniques*
6. *Laplace Transforms*

3.1.1 Direct Integration

Since a differential equation contains derivatives, it is sometimes possible to obtain a solution by integration. This process removes derivatives and provides arbitrary constants in the solution. For example, given equation 3.1:

$$\frac{dy}{dx} - x = +4$$

and rewriting to make each term integrable:

$$dy - xdx = 4dx$$

and integrating:

$$\int dy - \int xdx = \int 4dx$$

$$y - \frac{x^2}{2} = 4x + C$$

And, solving explicitly for *y*: $$y = \frac{x^2}{2} + 4x + C$$

where *C* is an arbitrary constant of integration. Unfortunately, application of the direct integration process fails to work in a majority of cases.

3.1.2 Separation of Variables

If direct integration fails for a first order differential equation, the next step is to try to separate the variables and then perform direct integration. When a differential equation can be put in the form:

$$f_1(x)dx + f_2(y)dy = 0$$

where one term contains functions of *x* and *dx* only, and the other term contains functions of *y* and *dy* only, the variables are said to be separated. A solution can then be obtained by direct integration:

$$\int f_1(x)dx + \int f_2(y)dy = C$$

where *C* is the arbitrary constant of integration. Note that for a differential equation of the first order, there is one arbitrary constant. In general, the number of arbitrary constants is equal to the order of the differential equation. Separation of variables is demonstrated with the following example:

$$\frac{dy}{dx} = \frac{x^2 + 3x + 4}{y + 6}$$

$$(y + 6)dy = (x^2 + 3x + 4)dx$$

$$\int (y + 6)dy = \int (x^2 + 3x + 4)dx$$

$$\frac{y^2}{2} + 6y = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + C$$

Not all first order differential equations can be separated in this fashion.

3.1.3 Exact Differential Integration

If direct integration or integration after separation of variables is not possible, then it still may be possible to obtain a solution *if the differential equation is an exact differential*. Associated with each suitably differentiable function of two variables *f* (*x*,*y*), there is an expression called its differential, namely:

$$df \equiv \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)dx + N(x,y)dy = 0 \tag{3.4}$$

This function is an exact differential if and only if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{3.5}$$

If this is so, then for all values of *C*:

$$\int_a^x M(x,y)dx + \int_b^y N(x,y)dy = C$$

is a solution to the differential equation. *a* and *b* are dummy variables of integration. If we take the example:

$$(2x + 3y - 2)dx + 3x - 4y + 1)dy = 0 \tag{3.6}$$

and apply the test in equation 3.5, we get:

$$\frac{\partial M}{\partial y} = \frac{\partial(2x + 3y - 2)}{\partial y} = 3$$

$$\frac{\partial N}{\partial x} = \frac{\partial(3x - 4y + 1)}{\partial x} = 3$$

Since the two partial derivatives are equal, the differential equation is exact. Its solution can be found by comparing equation 3.6 with equation 3.4 and realizing that:

$$\frac{\partial f}{\partial y} = 2x + 3y - 2 = 0 \tag{3.7}$$

$$\frac{\partial f}{\partial y} = 3x - 4y + 1 = 0 \tag{3.8}$$

Since equation 3.6 is an exact differential, then equations 3.7 and 3.8 can be obtained by taking partial derivatives of the same function *f* (*x*,*y*). To find the unknown function *f* (*x*,*y*), first integrate equations 3.7 and 3.8 assuming that y is constant when integrating with respect to *x* and that *x* is constant when integrating with respect to y:

$$f(x,y) = x^2 + 3xy - 2x + f(y) + C = 0 \tag{3.9}$$
$$f(x,y) = 3xy - 2y^2 + y + f(y) + C = 0 \tag{3.10}$$

Note that if equation 3.7 had been obtained from equation 3.9, any term that was a function of y only, and any constant term would have disappeared. Similarly, if equation 3.8 had been obtained from equation 3.10, the $f(x)$ and C terms would have likewise vanished. By a direct comparison of equations 3.9 and 3.10 the total function $f(x,y)$ can be determined:

$$f(x,y) = x^2 + 3xy - 2x - 2y^2 + y + \mathrm{C} = 0$$

Note that the unknown $f(y)$ term in equation 15.9 is $(-2y^2 + y)$ and the unknown $f(x)$ term in equation 3.10 is $(x^2 - 2x)$.

3.1.4 Integrating Factor

When none of the above procedures or techniques work, it may still be possible to integrate a differential equation using an integrating factor. When some unintegrable differential equation is multiplied by some algebraic factor which permits it to be integrated term by term, then the algebraic factor is called an integrating factor. Determining integrating factors for arbitrary differential equations is beyond the scope of this course; however, two integrating factors will be introduced in later sections of this chapter when developing operator techniques and Laplace transforms. These two factors will be e^{mx} and e^{-st}.

3.2 Operator Techniques

A form of differential equation that is of particular interest to us is:

$$A_n \frac{d^n x}{dt^n} + A_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + A_1 \frac{dx}{dt} + A_0 x = f(t) \tag{3.11}$$

If the coefficients $A_n, A_{n-1}, \cdots, A_0$ are functions of t only, then equation 3.11 is called a linear differential equation. If the coefficients are all constants, the equation 3.11 is called a linear differential equation with constant coefficients. Linear differential equations with constant coefficients occur frequently in the analysis of physical systems. Mathematicians and engineers have developed simple and effective techniques to solve this type of equation by using either "classical" or operational methods. When attempting to solve a linear differential equation of the form given in equation 3.11, it is helpful to first examine the equation:

$$A_n \frac{d^n x}{dt^n} + A_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + A_1 \frac{dx}{dt} + A_0 x = 0 \tag{3.12}$$

Equation 3.12 is the same as equation 3.11 with the right-hand side set equal to zero. Equation 3.11 is known as the general equation and 3.12 as the complementary or "homogeneous" equation. If the forcing function is zero, then the equation describes its inherent (natural) characteristics, i.e. with no external effects. Equation 3.11 may be interpreted as representing a physical system where the left-hand side of the equation describes the natural or designed state of the system, and where the right-hand side of the equation represents the input or forcing function. Solutions of equations 3.11 possess a useful property known as superposition, which may be briefly stated as follows: If $x_1(t)$ and $x2\ (t)$ are distinct solutions of equation 3.11, then any linear combination of $x_1(t)$ and $x_2(t)$, i.e., $C_1\, x_1\, (t) + C_2\, x_2\, (t)$, is also a solution. Shows that the general solution of equation 3.11 is then given by:

$$x(t) = x_c + x_p$$

where x_c is the complementary solution and x_p is the particular solution.

3.2.1 Complementary Solution

The solution to a **first order** linear differential equation can usually be obtained by direct integration. Consider the form:

$$\frac{dx}{dt} + mx = 0 \tag{3.13}$$

where m is a function of t only or a constant. To solve, we must separate the variables:

$$\frac{dx}{x} + m dt = 0$$

Integrating gives:

$$\int \frac{dx}{x} - \int m dt + C'$$

where $C' = \ln C$. Thus: $\ln(x) = -\, m dt + \ln(C)$

or: $x = Ce^{-\,mdt}$

If m is a constant, then: $x = Ce^{-\,mt}$ (3.14)

From this result, it can be concluded that a first order linear differential equation of the form of equation 3.13 can be solved by simply expressing the solution in the form of equation 3.14. To get a feel for this solution, note that at time zero, x is equal to C. As time grows to infinity, x approaches zero asymptotically. The rate of this approach (decay) is defined by magnitude of m. To better describe this decay rate for many engineering applications the Greek letter τ (tau) is introduced as the time constant.

$$\tau \equiv \frac{1}{m}$$

Thus, equation 3.14 could be rewritten as $x(t) = Ce^{-t/\tau}$

τ is the time, in seconds, required for x to decay to $1/e$ (= 0.368) of its initial value. As an example, consider the value for x in equation 15.14 after t seconds have elapsed:

$$x(t) = Ce^{-t/\tau} = Ce^{-\tau/\tau} = C\frac{1}{e} = 0.368C$$

The time history for such a first order homogeneous equation would look like Figure 3.1

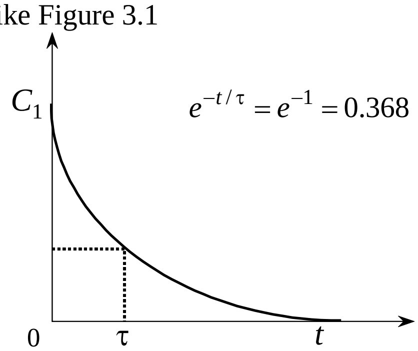


Figure 3.1 First Order Exponential Decay for $x(t) = Ce^{-t/\tau}$

Other measures of time are sometimes used to describe the decay of the exponential of a solution. If $T_{1/2}$ is used to denote the time it takes for the transient to decay to one-half of its initial amplitude, then:

$$T_{1/2} = 0.693\tau$$

This relationship can be easily shown by examining the equation:

$$x(t) = Ce^{mt}$$

By definition, τ□$= 1/m$. $T_{1/2}$ is the value of t at which $x = \frac{1}{2}x(0)$. The value of x at time zero is $x = Ce^{mt} = C$. Therefore the value at $t_{1/2}$ seconds is $x = \frac{C}{2}$.

$$\frac{1}{2}x_c(0) = \frac{1}{2}C = Ce^{-mT_{1/2}}$$

$$e^{-mT_{1/2}} = \frac{1}{2}$$

$$-mT_{1/2} = \ln\frac{1}{2}$$

$$T_{1/2} = -\frac{\ln\frac{1}{2}}{m} = \frac{0.693}{m} = 0.693\tau$$

The solution of a first order differential equation is always of exponential form; hopefully, solutions of higher order equations of the same family take the same form. Consider second order homogenous equation:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0 \quad (3.15)$$

If we assume the candidate solution:

$$x = Ce^{mt} \text{ then } \dot{x} = Cme^{mt},\ \ddot{x} = Cm^2e^{mt} \quad (3.16)$$

Substituting equation 3.16 into equation 3.15, gives:

$$C\,(am^2\,e^{mt} + bme^{mt} + ce^{mt}) = 0$$

or: $(am^2 + bm + c)e^{mt} = 0$

Since, $e^{mt} \neq 0$

then: $am^2 + bm + c = 0$

Recall the quadratic equation:

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic Equation}$$

Substituting these values into the assumed candidate solution of equation 3.16 gives the solution to equation 3.15:

$$x = c_1 e^{m_1 t} + c_2 e^{m_2 t} \tag{3.17}$$

Since there is no forcing function in equation 3.15, equation 3.17 represents both the complementary solution and the general solution as well.

Depending on the relative magnitudes of *a*, *b*, and *c*. In equation 3.15, there are four possibilities for m_1 and m_2, each of which is discussed below.

Case 1: Roots Real and Unequal. When $b^2 > 4ac$ the roots are real and unequal. The complementary solution has the form:

$$x = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

When m_1 and $m2$ are both negative, the system decays and there will be a time constant associated with each exponential term as shown in Figure 3.2.

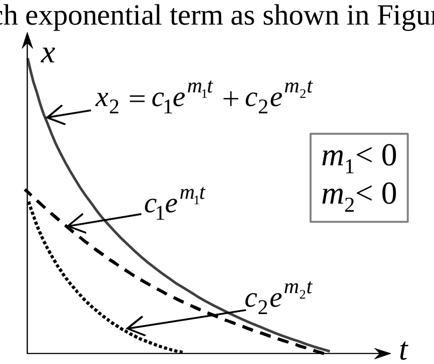


Figure 3.2 Second Order Transient Response with Real, Unequal, Negative Roots

When both roots are positive, the system will diverge as shown in Figure 3.4.

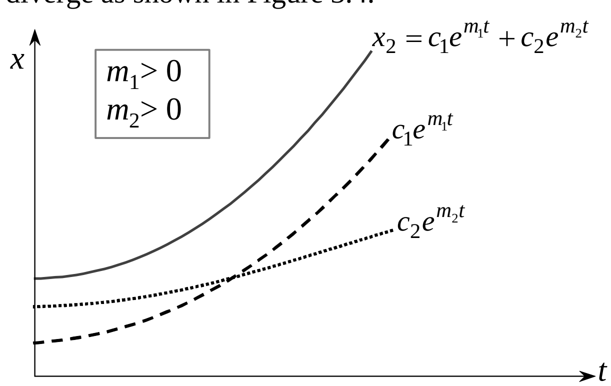


Figure 3.4 Second Order Transient Response with

When one root is positive and the other negative, the solution will eventually diverge as in Figure 3.3.

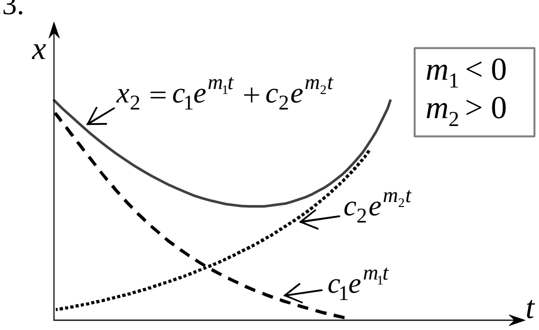


Figure 3.3 Second Order Transient Response with One Positive and One Negative Real, Unequal Root

An example of real and unequal roots case is given with the homogeneous differential equation:

$$\frac{d^2 x}{dt^2} + 4\frac{dx}{dt} - 12x = 0$$

Rewriting in operator form gives:

$$(m^2 + 4m - 12)e^{mt} = 0$$

Solving for the values of *m* gives:

$$m_{1,2} = \frac{-4 \pm \sqrt{16+48}}{2} = \frac{-4 \pm 8}{2} = -6{,}2$$

the solution is: $x = c_1 e^{-6t} + c_2 e^{2t}$

Two Real, Unequal, Positive Roots

Case 2: Roots Real and Equal. If $b^2 = 4ac$, then m_1 and m_2 are real and equal, and an alternate form of solution is required. Given the homogeneous differential equation:

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0 \tag{3.18}$$

Rewriting in operator form gives: $(m^2 - 4m + 4)e^{mt} = 0$

Solving for the values of m gives:

$$m_{1,2} = \frac{-4 \pm \sqrt{16-16}}{2} = \frac{4}{2} = 2$$

But this only gives one distinct value of m, and two values of m are required to result in a solution of the form of equation 3.17 which has two arbitrary constants.

> Writing the solution in the form of equation 3.17 when the roots are repeated does not give a solution because the two arbitrary constants can be combined into a single arbitrary constant as shown below:
>
> $$x = c_1 e^{2t} + c_2 e^{2t} = (c_1 + c_2)e^{2t} = c_3 e^2$$

To solve this problem, one of the arbitrary constants must simply be multiplied by t (remember, we said that the coefficients could be functions of the independent variable). The solution now contains two arbitrary constants which cannot be combined, and it is easily verified that:

$$x = c_1 e^{2t} + c_2 te^{2t}$$

is a solution of equation 3.18.

When m is negative, the system will usually decay as shown in Figure 3.5. If m is very small, the system may initially exhibit divergence. When m is positive, the system will diverge.

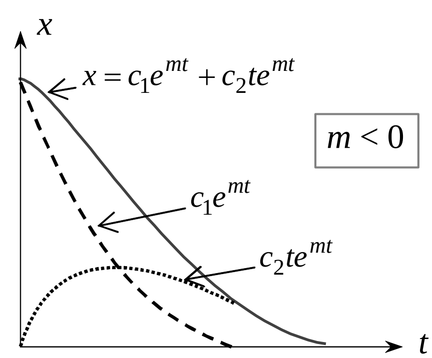


Figure 3.5 Second Order Transient Response with Real, Equal, Negative Roots

3.2.1.1 Case 3: Roots Purely Imaginary.

When b (the damping coefficient) is zero in equation 3.5, the roots are purely imaginary. Given the homogeneous differential equation:

$$\frac{d^2x}{dt^2} + x = 0$$

Rewriting in operator form gives: $(m^2 + 1)e^{mt} = 0$

Solving for the values of m gives:

$$m_{1,2} = \frac{-0 \pm \sqrt{0-4}}{2} = \pm\sqrt{-1} = \pm j$$

In most engineering work $\sqrt{-1}$ is given the symbol j. The solution is written:

$$x = c_1 e^{jt} + c_2 e^{-jt}$$

This is a perfectly good solution from a mathematical standpoint, but, Euler's identity can be used to put the solution in a more usable form:

$$e^{jt} = \cos t + j \sin t \quad \text{Euler's Identity}$$

This equation can be restated in many ways geometrically and analytically, and be verified by adding the series expansion of cos x to the series expansion of j sin x. The complementary solution may be expressed as follows:

$$x = c_1 \; [\cos t \; + j \sin t] + c_2 \, [\cos (-t) + j \sin (-t) \,]$$
$$x = (c_1 + c_2)\cos t + j(c_1 - c_2)\sin t$$
$$x = c_3 \cos t + c_4 \sin t \qquad (3.19)$$

An equivalent expression may be:

$$x_c = \sqrt{c_3^2 + c_4^2}\left[\frac{c_3}{\sqrt{c_3^2 + c_4^2}}\cos t + \frac{c_4}{\sqrt{c_3^2 + c_4^2}}\sin t\right]$$

if the arbitrary constants c_3 and c_4 are related as shown in Figure 3.6.

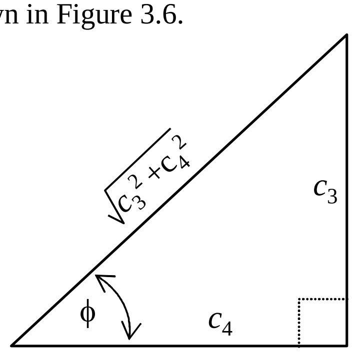


Figure 3.6 Definition of c_3 and c_4

Then:

$$\frac{c_3}{\sqrt{c_3^2 + c_4^2}} = \sin \phi$$

$$\frac{c_4}{\sqrt{c_3^2 + c_4^2}} = \cos \phi$$

$$\frac{c_4}{\sqrt{c_3^2 + c_4^2}} = A$$

where A and ϕ are now the two arbitrary constants. The solution now becomes:

$$x = A \,(\sin \phi \cos t + \cos \phi \sin t) = A \,(\sin t + \phi) \qquad (3.20)$$

Note that the above equation could also be written in the equivalent form:

$$x = A(\cos \theta \cos t + \sin \theta \sin t) = A \cos (t - \theta) \qquad (3.21)$$

where $\theta = 90° - \phi$.

In summary, the following solutions are equivalent:

$$x = c_1 e^{jt} + c_2 e^{-jt}$$
$$x = c_1 \sin kt + c_2 \cos kt$$
$$x = A \sin (kt + \phi)$$
$$x = A \cos (kt - \theta)$$

The system exhibits periodic oscillations of constant amplitude with a frequency k as shown in Figure 3.7.

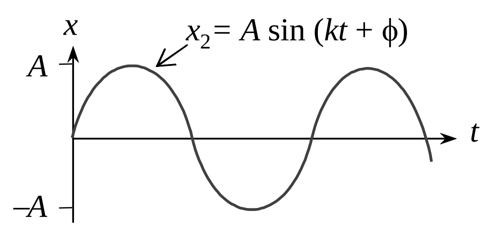


Figure 3.7 Second Order Transient Response with Pure Imaginary Roots

Case 4: Roots Complex Conjugates. When the coefficients of equation 3.15 are such that $4ac > b^2$ and $b \neq 0$, then the roots are complex conjugates. Given the homogeneous differential equation:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$$

Rewriting in operator form gives:

$$(m^2 + 2m + 2)e^{mt} = 0$$

Solving for the values of m gives:

$$m_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \sqrt{-1} = -1 + j, -1 - j$$

These roots are complex conjugates and give the solution:

$$x = c_1 e^{(-1+j)t} + c_2 e^{(-1-j)t} = e^{-t}\left(c_1 e^{jt} + c_2 e^{-jt}\right)$$

or, using the results from equations 3.19, 3.20 and 3.21, alternate solutions can immediately be written as:

$$x = e^{-t}(c_3 \cos t + c_4 \sin t) = Ae^{-t} \sin(t + \phi) = Ae^{-t} \cos(t - \phi) \qquad (3.22)$$

In more general terms, when the roots are given by $m_{1,2} = k_1 \neq jk_2$, the complementary solution has one of the following three forms:

$$x = e^{k_1 t}\left(c_1 \cos k_2 t + c_2 \sin k_2 t\right)$$

$$x = Ae^{k_1 t} \sin\left(k_2 t + \phi\right)$$

$$x = Ae^{k_1 t} \cos\left(k_2 t - \phi\right)$$

The system exhibits periodic oscillations bounded within an envelope given by $x = \pm e^{k_1 t}$. When k_1 is negative, the system oscillations decay or converge as shown in Figure 3.8. When k_1 is positive, the system oscillations diverge as shown in Figure 3.9.

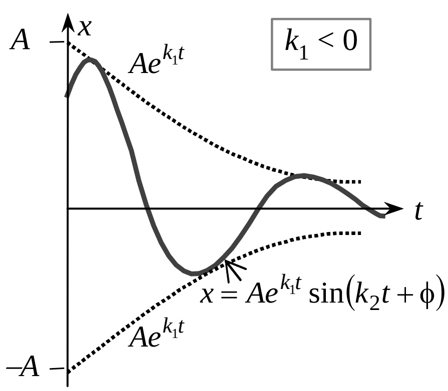


Figure 3.8 Second Order Convergent Transient Response with Complex Conjugate Roots

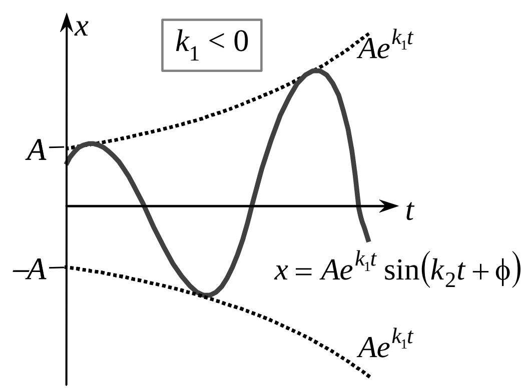


Figure 3.9 Second Order Divergent Transient Response with Complex Conjugate Roots

The above discussion of solutions (responses) reveals only part of the picture introduced by equation 3.11; the input or forcing function $f(t)$ is still left to consider. In practice, a linear system that is inherently divergent (without input) may be damped by carefully selecting or controlling the input. Conversely, a convergent linear system may be driven divergent by certain types of inputs. Linear control theory examines these problems in more detail.

3.2.2 Solved Problems

A few solved problems will illustrate some response cases. For all examples below, the forcing function is zero, therefore the (complementary) solution is sufficiently descriptive.

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f(t)$$

where each term has the same qualitative significance but different physical significance.

Example 1: Given the homogeneous linear differential equation:

$$\ddot{x} + 4x = 0$$

The system is undamped with a complementary solution:

$$x = A \sin(2t + \phi)$$

where A and ϕ are constants of integration which could be determined by substituting boundary conditions into the general solution.

Example 2: Given the homogeneous linear differential equation:

$$\ddot{x} + \dot{x} + x = 0$$

The system is underdamped with a complementary solution:

$$x = A\mathrm{e}^{-0.5t} \sin(0.866t + \phi)$$

Example 3: Given the homogeneous linear differential equation:

$$\ddot{x} + 4\dot{x} + 4x = 0$$

The system is critically damped with a complementary solution:

$$x = c_1 e^{-2t} + c_2 t e^{-2t}$$

Example 4: Given the homogeneous linear differential equation:

$$\ddot{x} + 8\dot{x} + 4x = 0$$

The system is overdamped with a complementary solution:

$$x = c_1 e^{-7.46t} + c_2 e^{-0.54}$$

Example 5: Given the homogeneous linear differential equation:

$$\ddot{x} - 2\dot{x} + 4x = 0$$

The system is unstable with a complementary solution:

$$x = Ae^{t} \sin(1.732t + \phi)$$

3.3 Particular Solution

Up to this point, all examples illustrated "natural" system responses with no forcing function present. The principle of superposition states that, if some forcing function is applied, the total response is simply the sum of the natural response plus the forced response due to that particular input. This forced response is known as the particular or "steady-state" solution. The particular solution to a linear differential equation can be obtained by the method of undetermined coefficients. This method consists of assuming a solution of the same general form as the input (forcing function), but with undetermined constant coefficients. Substitution of this assumed solution into the differential equation enables the coefficients to be evaluated. The method of undetermined coefficients applies when the forcing function or input is polynomial, transcendental, exponential, or combinations of sums and products of these terms. The general solution to the differential equation with constant coefficients is then given by:

$$x(t) = x_c + x_p$$

which is the summation of the solution to the homogeneous equation (complementary solution), plus the particular solution.

Consider the linear differential equation:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

The particular solution which results from a given input, $f(t)$, can be solved for using the method of undetermined coefficients as illustrated in the following examples.

3.3.1 Constant Forcing Function

Given the linear differential equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 6 \tag{3.23}$$

The input is a constant, so a solution of the form $x_p = K$ is assumed. Then:

$$\frac{dx_p}{dt} = \frac{d}{dt}K = 0$$

$$\frac{d^2x_p}{dt^2} + 4\frac{d^2}{dt^2}K = 0$$

Substituting into equation 3.23 gives:

$$0 + 4(0) + 3K = 6$$
$$x_p = K = 2$$

Therefore, $x_p = 2$ is a particular solution. The homogeneous equation can be solved using the operator technique to yield the complementary solution. Note that the subscript "c" is now used to delineate between the general solution, $x(t)$, and the complementary solution, x_c.

$$x_c = c_1 e^{-t} + c_2 e^{-3t}$$

The general solution is

$$x(t) = x_c + x_p = c_1 e^{-t} + c_2 e^{-3t} + 2$$

3.3.2 Polynomial Forcing Function

Given the linear differential equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = t^2 + 2t \tag{3.24}$$

The form of $f(t)$ for equation 3.24 is a polynomial of second degree, so a particular solution of second degree is assumed:

$$x_p = A t^2 + B t + C$$

Then:

$$\frac{dx_p}{dt} = 2At + B$$

$$\frac{d^2x_p}{dt^2} = 2A$$

Substituting into equation 3.24 gives:

$$(2A) + 4(2At + B) + 3(At_2 + Bt + C) = t^2 + 2t$$
$$(3A)t_2 + (8A + 3B)t + (2A + 4B + 3C) = t^2 + 2t$$

Equating like powers of t and solving for the constants gives:

$$3A = 1$$

$$A = \frac{1}{3}$$

$$8A + 3B = 2$$

$$B = -\frac{2}{9}$$

$$2A + 4B + 3C = 0$$

$$C = \frac{2}{27}$$

Therefore: $$x_p = \tfrac{1}{3}t^2 - \tfrac{2}{9}t + \tfrac{2}{27}$$

The general solution is then the previous homogenous solution plus this new particular solution:

$$x(t) = \left[c_1e^{-t} + c_2e^{-3t}\right] + \left[\tfrac{1}{3}t^2 - \tfrac{2}{9}t + \tfrac{2}{27}\right]$$

As a general rule, if the forcing function is a polynomial of degree *n*, assume a polynomial solution of degree *n*.

3.3.3 Exponential Forcing Function

Given the linear differential equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = e^{2t} \tag{3.25}$$

The form of the forcing function for equation 3.25 is exponential, so assuming an exponential particular solution gives:

$$x_p = Ae^{2t}$$

Then: $$\frac{dx_p}{dt} = 2Ae^{2t}$$

$$\frac{d^2x_p}{dt^2} = 4Ae^{2t}$$

Substituting into equation 3.25 gives:

$$(4Ae^{2t}) + 4(2Ae^{2t}) + 3(Ae^{2t}) = e^{2t}$$
$$(4A + 8A + 3A)e^{2t} = e^{2t}$$

Equating the coefficients and solving for the constants gives:

$$15A = 1$$

$$A = \frac{1}{15}$$

Therefore: $$x_p = \frac{1}{15}e^{2x}$$

and the general solution is then: $$x(t) = c_1e^{-t} + c_2e^{-3t} + \frac{1}{15}e^{2t}$$

A final example will illustrate a pitfall sometimes encountered using the method of undetermined coefficients.

3.3.4 Exponential Forcing Function (special case)

Given the linear differential equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = e^{-t} \tag{3.26}$$

The forcing function is e^{-t}, so assuming a solution of the form:

$$x_p = Ae^{-t}$$

Then: $$\frac{dx_p}{dt} = -Ae^{-t}$$

$$\frac{d^2x_p}{dt^2} = Ae^{-t}$$

Substituting into equation 3.26 gives:

$$Ae^{-t} - 4Ae^{-t} + 3Ae^{-t} = e^{-t}$$

$$(A - 4A + 3A)e^{-t} = e^{-t}$$
$$(0)e^{-t} = (1)e^{-t}$$

Obviously, this is an incorrect statement. To solve the equation, as before when we encountered this sort of dichotomy, we instead, assume a solution of the form:

$$x_p = Ate^{-t}$$

Then:
$$\frac{dx_p}{dt} = A\left(e^{-t} - te^{-t}\right)$$
$$\frac{d^2x_p}{dt^2} = A\left(te^{-t} - 2e^{-t}\right)$$

Substituting into equation 3.26 gives:

$$A(te^{-t} - 2e^{-t}) + 4A(e^{-t} - te^{-t}) + 3Ate^{-t} = e^{-t}$$
$$(A - 4A + 3A)te^{-t} + (-2A + 4A)e^{-t} = e^{-t}$$
$$(0)te^{-t} + 2Ae^{-t} = e^{-t}$$

Equating the coefficients and solving for the constants gives:

$$2A = 1$$
$$A = \frac{1}{2}$$

which will lead to the solution:

$$x(t) = x_c + x_p = c_1e^{-t} + c_2e^{-3t} + \tfrac{1}{2}te^{-t}$$

The key to successful application of the method of undetermined coefficients is to assume the proper form for a candidate particular solution.

3.4 Solving for Constants of Integration

The number of arbitrary constants in the solution of a linear differential equation is equal to the order of the equation. The constants of integration can be determined by initial conditions or boundary conditions. That is, to solve for the constants of integration, the physical state of the system must be known at some time. The number of initial or boundary conditions given must be equal to the number of constants to be solved for. Many times these conditions are given at time equal to zero, in which case they are called initial conditions. A system which has zero initial conditions, i.e., initial position, velocity and acceleration all equal to zero, is called a quiescent system.

The arbitrary constants of the solution must be evaluated from the general solution; that is, the transient (or homogeneous or complementary) solution plus the steady-state (particular) solution. The method of evaluating the constants of integration will be illustrated with an example. Given the linear differential equation:

$$\ddot{x} + 4\dot{x} + 13x = 3 \qquad (3.27)$$

Because this is a second order differential equation, 2 initial conditions are required. For this example, assume

$$x(0) = 5$$

and
$$\dot{x}(0) = 8$$

The complementary solution is:
$$m^2 + 4m + 13 = 0$$
$$m_{1,2} = -2 \pm \sqrt{4 - 13} = -2 \pm j3$$
$$x_c = (A\cos 3t + B\sin 3)e^{-2t}$$

Assuming a particular solution of the form:

$$x_p = C$$
$$\dot{x}_p = 0$$

$$\ddot{x}_p = 0$$

Substituting into equation 3.27 gives:

$$(0) + 4(0) + 13C = 3$$

$$C = \frac{3}{13}$$

The general solution is then:

$$x(t) = (A\cos 3t + B\sin 3t)e^{-2t} + \frac{3}{13}$$

To solve for *A* and *B*, the initial conditions specified above are used:

$$x(0) = 5 = A + \frac{3}{13}$$

$$A = \frac{62}{13}$$

Differentiating the general solution:

$$\dot{x}(t) = (-3A\sin 3t + 3B\cos 3t)e^{-2t} - 2e^{-2t}(A\cos 3t + B\sin 3t)$$

and substituting the second initial condition gives:

$$\dot{x}(0) = 8 = 3B - 2A$$

$$B = \frac{76}{13}$$

Therefore, the complete solution to equation 3.27 with the given initial conditions is:

$$x(0) = \left(\frac{62}{13}\cos 3t + \frac{76}{13}\sin 3t\right)e^{-2t} + \frac{3}{13}$$

The principle also applies to the first order systems. Consider the first order linear differential equation:

$$4\dot{x} + x = 3 \qquad (3.28)$$

Physically, *x* can represent distance or displacement and *t* is used to represent time. The transient solution can be found from the homogeneous equation:

$$4\dot{x} + x = 0$$

$$(4m + 1)e^{mt} = 0$$

$$4m + 1 = 0$$

$$m - \frac{1}{4}$$

Thus: $x_c = ce^{-t/4}$

The particular solution is found by assuming:

$$x_p = A$$

Substituting this back into equation 3.28 gives:

$$A = 3$$

$$x_p = 3$$

which gives the general solution:

$$x(t) = ce^{-t/4} + 3 \qquad (3.29)$$

The first term on the right-hand side of equation 3.29 represents the transient response of the physical system described by the model equation 3.28; the second term represents the steady-state response of the system if the transient decays. The complete solution to equation 3.27 can be completed by specifying a boundary condition and evaluating the arbitrary constant. Letting $x = 0$ at $t = 0$ gives:

$$x(t) = ce^{-t/4} + 3$$

$$x(0) = 0 = c + 3$$

$$c = -3$$

The complete solution for this boundary condition then is:

$$x(t) = -3e^{-t/4} + 3$$

This is an exponential rise from 0 towards 3 as t approaches infinity.

First and second order differential equations have been discussed in some detail. It is of great importance to note that many higher order systems quite naturally decompose into first and second order parts. This is handled by factoring and solving the characteristic equation describing the system to get a transient and steady-state solution by any convenient method. This will become clearer in the later portions of this chapter. One final remark is appropriate regarding the second order linear differential equation with constant coefficients. Although the equation is interesting in its own right, it is of particular value because it is a mathematical model for several problems of physical interest which will be addressed later.

3.5 Laplace Transforms

A technique has been presented for solving linear differential equations with constant coefficients, with and without inputs or forcing functions. The method has limitations since it is suited for differential equations with inputs of only certain forms. Some procedures require looking for special cases which require careful handling but operator procedures have the remarkable property of changing or "transforming" a problem of integration into a problem in algebra; i.e., solving a quadratic equation in the case of linear second order differential equations. This is accomplished by making an assumption involving the number e.

There is another technique which exchanges (transforms) the whole differential equation, including the input and initial conditions into an algebra problem. Fortunately, the method applies to linear first and second order differential equations with constant coefficients. If we multiply each term in equation 3.11 by the integrating factor e^{mt}, we get:

$$a\ddot{x}(t)e^{mt} + b\dot{x}(t)e^{mt} + cx(t)e^{mt} = f(t)e^{mt} \tag{3.30}$$

It is now possible that equation 3.30 can be integrated term by term on both sides of the equation to produce an algebraic expression in m. The algebraic expression can then be manipulated to eventually obtain the solution of equation 3.11.

The new integrating factor e^{mt} should be distinguished from the previous integrating factor used in developing the operator techniques for solving the homogeneous equation. In order to accomplish this, m will be replaced by $-s$. The reason for the minus sign will become clear later. In order to integrate the terms in equation 3.30, limits of integration are required. In most physical problems, events of interest take place subsequent to a given starting time which we call $t = 0$. To be sure to include the duration of all significant events, the composite of effects from time $t = 0$ and $\tau = \infty$ will be included. Equation 3.30 now becomes:

$$\int_0^\infty a\ddot{x}(t)e^{-st}\,dt + \int_0^\infty b\dot{x}(t)e^{-st}\,dt + \int_0^\infty cx(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-st} \tag{3.31}$$

Equation 3.31 is called the Laplace transform of equation 3.11. The problem now is to integrate the terms in the equation.

3.5.1 Finding the Laplace Transform of a Differential Equation

The Laplace transform is defined as:

$$L\{x(t)\} \equiv X(s) \equiv \int_0^\infty x(t)e^{-st}\,dt \tag{3.32}$$

where the letter L is used to signify a Laplace transform. $x(s)$ must, for the present, remain an unknown (previously m was carried along as an unknown until the auxiliary equation evolved, at which time m was solved for explicitly). Since equation 3.32 transforms $x(t)$ into a function of the variable s, then:

$$L\{cx(t)\} \equiv \int_0^\infty cx(t)e^{-st}\,dt = c\int_0^\infty x(t)e^{-st}\,dt = cX(s)$$

The transform for the second term is given by:

$$L\{b\dot{x}(t)\} \equiv \int_0^\infty b\dot{x}(t)e^{-st}\,dt = b\int_0^\infty \dot{x}(t)e^{-st}\,dt$$

To solve this equation, integration by parts is used:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

To apply this formula let:

$$u = e^{-st}$$
$$dv = \dot{x}(t)dt$$
$$du = -se^{-st}\,dt$$
$$v = x(t)$$

Substituting these values and integrating from t = 0 to $\tau = \infty$:

$$\int_0^\infty \dot{x}(t)e^{-st}\,dt = \left[x(t)e^{-st}\right]_0^\infty - \int_0^\infty x(t)\left(-se^{-st}\right)dt$$

$$= \left[x(t)e^{-st}\right]_0^\infty + s\int_0^\infty x(t)e^{-st}\,dt$$

Therefore,

$$\int_0^\infty \dot{x}(t)e^{-st}\,dt = \left[x(t)e^{-st}\right]_0^\infty + sX(s) + \lim_{t\to\infty} x(t)e^{-st} - (0)$$

and assume that the term e^{-st} "dominates" the term $x(t)$ as $t \to \infty$. The reason for using the minus sign in the exponential should now be apparent. Thus, $\lim_{t\to\infty} x(t)e^{-st} = 0$, and the above equation becomes:

$$L\{b\dot{x}(t)\} \equiv \int_0^\infty b\dot{x}(t)e^{-st}\,dt = b[sX(s) - x(0)]$$

This equation can be extended to higher order derivatives. Without showing the full details of the derivation, such an extension yields:

$$L\{b\ddot{x}(t)\} \equiv a\left[s^2X(s) - sx(0) - \dot{x}(0)\right]$$

Returning to equation 3.31, note that the Laplace transforms of all terms except the forcing function have been found. To solve this transform, the forcing function must be specified. A few typical forcing functions will be considered to illustrate the technique for finding Laplace transforms. If the forcing function were a constant:

$$f(t) = A$$

Then:

$$L\{A\} \equiv \int_0^\infty Ae^{-st}\,dt = -\frac{A}{s}\int_0^\infty e^{-st}(-s\,dt) = -\frac{A}{s}\left[e^{-st}\right]_0^\infty = \frac{A}{s}$$

If the forcing function were a polynomial in t:

$$f(t) = t$$

Then:

$$L\{t\} \equiv \int_0^\infty te^{-st}\,dt$$

To solve, we must integrate by parts, let:

$$u = t$$
$$dv = e^{-st}dt$$
$$du = dt$$

Therefore:

$$L\{t\} \equiv \int_0^\infty te^{-st}\,dt = \left[-\frac{1}{s}te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = 0 + \frac{1}{s^2} = \frac{1}{s^2}$$

If the forcing function were an exponential:

$$f(t) = e^t$$

Then: $$L\{e^t\} \equiv \int_0^\infty e^t e^{-st}\,dt = \int_0^\infty e^{(1-s)t}\,dt = \frac{1}{s-1}$$

The Laplace transforms of more complicated functions may be quite tedious to derive, but the procedure is similar to that above. Fortunately, it is not necessary to derive Laplace transforms each time they are needed. Extensive tables of transforms exist in most advanced mathematics and linear control system textbooks. All of the transforms needed for this course are listed in Table 3.1.

	Laplace Transforms	
1	$X(s)$	$x(t)$
2	$a[sX(s)-x(0)]$	$a\dot{x}(t)$
3	$a[s^2X(s)-sx(0)-\dot{x}(0)]$ (can be extended to any necessary order)	$a\ddot{x}(t)$
4	$1/s$	1
5	$1/s^2$	t
6	$\frac{n!}{s^{n+1}}(n=1,2,\ldots)$	t^n
7	$\frac{1}{s+a}$	e^{-at}
8	$\frac{1}{s+a^2}$	te^{-at}
9	$\frac{n!}{(s+a)^{n+1}}(n=1,2,\ldots)$	t^ne^{-at}
10	$\frac{1}{(s+a)(s+b)}(a\neq b)$	$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$
11	$\frac{s}{(s+a)(s+b)}(a\neq b)$	$\frac{1}{a-b}\left(ae^{-at}-ae^{-bt}\right)$
12	$\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{(b-c)e^{-at}-(a-c)e^{-bt}+(a-b)e^{-ct}}{(a-b)(b-c)(a-c)}$
13	$\frac{a}{s^2+a^2}$	$\sin at$
14	$\frac{s}{s^2+a^2}$	$\cos at$
15	$\frac{a^2}{s(s^2+a^2)}$	$1-\cos at$
16	$\frac{a^3}{s^2(s^2+a^2)}$	$at-\sin at$
17	$\frac{2a^3}{(s^2+a^2)^2}$	$\sin at - at\cos at$
18	$\frac{2as}{(s^2+a^2)^2}$	$t\sin at$
19	$\frac{2as^2}{(s^2+a^2)^2}$	$\sin at + at\cos at$
20	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
21	$\frac{(b^2-a^2)s}{(s^2+a^2)(s^2+b^2)}(a^2\neq b^2)$	$\cos at - \cos bt$
22	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
23	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$

Table 3.1 Laplace Transforms

The technique of using Laplace transforms to assist in the solution of a differential equation is best described in an example:

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t} \tag{3.33}$$

with the initial conditions:

$$x(0) = 1$$
$$\dot{x}(0) = -4$$

Taking the Laplace transform of equation 3.33 gives:

$$s^2 X(s) - sx(0) - \dot{x}(0) + 4[sX(x) - x(0) + 4X(s)] = \frac{4}{s-2}$$

Solving for $X(s)$:

$$X(s) = \frac{s^2 - 2s + 4}{(s-2)(s+2)^2} \tag{3.34}$$

If we have a transform of this type in the table then it could be used to reverse the Laplace operation to find the solution. In order to continue with the solution in this case however, it is necessary to discuss partial fraction expansions.

3.5.2 Partial Fraction Expansion

The method of partial fractions enables the separation of a complicated rational proper fraction into a sum of simpler fractions. If the fraction is not proper (the degree of the numerator less than the degree of the denominator), it can be made proper by dividing the fraction and considering the remaining expression. Given a fraction of two polynomials in the variable s as shown in equation 3.34, there occur several cases:

Case 1: Distinct Linear Factors. To each linear factor such as $(as + b)$ occurring in the denominator, there corresponds a single partial fraction of the form $A/(as + b)$, e.g.:

$$\frac{7s-4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

where *A*, *B*, and *C* are constants to be determined.

Case 2: Repeated Linear Factors. To each linear factor, $(as + b)$, occurring *n* times in the denominator, there corresponds a set of *n* partial fractions, e.g.:

$$\frac{s^2 - 9s + 17}{(s-2)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

where *A*, *B*, and *C* are constants to be determined.

Case 3: Distinct Quadratic Factors. To each irreducible quadratic factor, $(as^2 + bs + c)$, occurring in the denominator, there corresponds a single partial fraction of the form $(As + B)/$, e.g.:

$$\frac{3s^2 + 5s + 8}{(s+2)(s^2+1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

where *A*, *B*, and *C* are constants to be determined.

Case 4: Repeated Quadratic Factors. To each irreducible quadratic factor, $(as^2 + bs + c)$, occurring *n* times in the denominator, the corresponds a set of *n* partial fractions, e.g.:

$$\frac{10s^2+s+36}{(s-4)(s^2+4)^2}=\frac{A}{s-4}+\frac{Bs+C}{s^2+4}+\frac{Ds+E}{(s^2+4)^2}$$

where *A*, *B*, *C*, *D*, and *E* are constants to be determined.

The "brute force" technique for finding the constants is the same method of undetermined coefficients we used earlier and will be illustrated by solving the equation above for case 4. We start by finding the common denominator on the right side of the equation:

$$\frac{10s^2+s+36}{(s-4)(s^2+4)^2}=\frac{A(s^2+4)^2+(Bs+C)(s-4)(s^2+4)+(Ds+E)}{(s-4)(s^2+4)^2}$$

The numerators are then set equal to each other:

$$10s^2+s+36=A(s^2+4)^2+(Bs+C)(s-4)(s^2+4)+(Ds+E)(s-4)$$

Since this equation must hold for all values of s, enough values of *s* are simply substituted into the equation to find the five constants. The values we usually start with are the roots of the denominator terms; i.e., $s = 4, 2j, -2j$. When we do this, we get $A = 1/2$, $B = -1/2$, $C = -2$, $D = 0$, and $E = 1$. When we substitute these values back into the initial equation, we get:

$$\frac{10s^2+s+36}{(s-4)(s^2+4)^2}=\frac{1}{2(s-4)}-\frac{(s-4)}{2(s^2+4)}+\frac{1}{(s^2+4)^2}$$

Returning now to the example Laplace transform solution equation 3.34 which can be expanded by partial fractions to get:

$$X(s)\frac{s^2-2s+4}{(s-2)(s+2)^2}=\frac{A}{s-2}+\frac{B}{s+2}+\frac{C}{(s+2)^2} \tag{3.35}$$

Taking the common denominator and setting the numerators equal:

$$s^2-2s+4=A(s+2)^2+B(s+2)(s-2)+C(s-2)$$

The "brute force" method illustrated above could again be employed to solve for the constants *A*, *B*, and *C*. An alternate method exists, however, to solve for the constants. Multiplying out the right side of the equation and collecting coefficient terms for each power of s gives:

$$s^2-2s+4=(A+B)s\ 2+(4A+C)s+(4A-4B-2C)$$

Now the coefficients of like powers of s on both sides of the equation must be equal. Equating gives:

$$s^2:\ 1=A+B$$
$$s^1:\ -2=4A+C$$
$$s^0:\ 4=4A-4B-2C$$

Solving for the constants gives:

$$A=1/4$$
$$B=3/4$$
$$C=-3$$

Substituting the constants into equation 3.35 results in:

$$X(s)=\frac{1}{4(s-2)}+\frac{3}{4(s+2)}+\frac{3}{(s+2)^2} \tag{3.36}$$

Another expansion method called the Heaviside Expansion Theorem can be used to solve for constants in the numerator of distinct linear factors (it doesn't work for quadratic factors). This method of expansion is used extensively in linear control theory. If the denominator of an expansion term has a distinct linear factor, $(s-a)$, the constant for the factor can be found by multiplying $X(s)$ by $(s-a)$ and evaluating the remainder of $X(s)$ at $s = a$. Using an example to illustrate:

$$X(s) = \frac{7s-4}{s(s-1)(s-2)} = \frac{A}{s}\frac{B}{s-1} + \frac{C}{s+2}$$

$$A = [sX(s)]_{s=0} = \left[\frac{7s-4}{(s-1)(s+2)}\right]_{s=0} = \frac{-4}{(-1)(2)} = 2$$

$$B = [(s-1)X(s)]_{s=1} = \left[\frac{7s-4}{s(s+2)}\right]_{s=0} = \frac{7-4}{(1)(3)} = 2$$

$$C = [(s+2)X(s)]_{s=2} = \left[\frac{7s-4}{s(s-1)}\right]_{s=2} = \frac{-14-4}{(-2)(-3)} = -3$$

3.5.3 The Inverse Laplace Transform

Now that the methods to expand the right-hand side of $X(s)$ have been discussed in detail, all that remains is to transform the expanded terms back to the time domain. This is easily accomplished using a Laplace transform table. Returning to equation 3.36 and using Table 3.1, the equation can be easily transformed to time domain solution:

$$x(t) = \tfrac{1}{4}e^{2t} + \tfrac{3}{4}e^{-2t} - 3te^{-2}$$

In summary, the strength of the Laplace transform is that it converts linear differential equations with constant coefficients into algebraic equations in the *s*-domain. All that remains is to take the inverse transform of the explicit solutions to return to the time domain. Although the applications in this text will consider only time as the independent variable, a linear differential equation with any independent variable may be solved by Laplace transforms.

3.5.4 Laplace Transform Properties

There are several important properties of the Laplace transform which should be included in the ongoing discussion. In the general case:

$$L\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - \left[s^{n-1}x(0) + s^{n-2}\frac{dx(0)}{dt} + \cdots + \frac{d^{n-1}x(0)}{dt^{n-1}}\right]$$

For quiescent systems:

$$L\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s)$$

This result enables transfer functions to be written by inspection. For example, given the linear differential equation:

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t}$$

With quiescent initial conditions, the Laplace transform can immediately be written by inspection as:

$$\left(s^2 + 4s + 4\right)X(s) = \frac{4}{s-2}$$

Another significant transform is that of an indefinite integral. In the general case:

$$L\left\{\int\int\int\cdots x(t)dt^n\right\} = \frac{X(s)}{s^n} + \frac{1}{s^n}\left[\int x(t)dt\right]_{t=0} + \frac{1}{s^{n-1}}\left[\int\int x(t)dt\right]_{t=0} + \cdots$$

This equation allows the transformation of integral-differential equations such as those arising in electrical engineering. For the case where all integrals of $f(t)$ evaluated at $t = 0$ are zero (quiescent system), the transform becomes:

$$L\left\{\int\int\int\cdots x(t)dt^n\right\} = \frac{X(s)}{s^n}$$

For example, given the linear differential equation:

$$\ddot{x} + 4\dot{x} + 4x\int x\,dt = 4e^{2t}$$

With quiescent initial conditions, the Laplace transform can immediately be written by inspection as:

$$\left(s^2 + 4s + 4 + \frac{1}{s}\right)X(s) = \frac{4}{s-2}$$

Multiplying both sides of the equation by s gives:

$$\left(s^3 + 4s^2 + 4s + 1\right)X(s) = \frac{4s}{s-2}$$

which raises the order of the left-hand side and acts to differentiate the right-hand side. The usefulness of the Laplace transform technique can best be demonstrated by several example problems.

Example 1: Given the linear differential equation:

$$\dot{x} + 2x = 1$$

with the initial conditions $x(0) = 1$. By Laplace transform:

$$\left[sX(s) - x(0)\right] + 2\left[X(s)\right] = \frac{1}{s}$$

Substituting initial conditions and solving for X(s):

$$(s+2)X(s) = 1 + \frac{1}{s}$$

$$X(s) = \frac{s+1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

Solving for the constants A and B:

$$s + 1 = A(s + 2) + Bs$$
$$s + 1 = (A + B)s + 2A$$
$$A = ½ \qquad B = ½$$

Thus:

$$X(s) = \frac{1}{2s} + \frac{1}{2(s+2)}$$

And, the inverse Laplace transform gives the final solution of:

$$x(t) = \tfrac{1}{2} + \tfrac{1}{2}e^{-2t}$$

Example 2: Given the linear differential equation:

$$\dot{x} + 2x = \sin t$$

with the initial conditions $x(0) = 5$. By Laplace transform:

$$\left[sX(s) - x(0)\right] + 2\left[X(s)\right] = \frac{1}{s^2+1}$$

Substituting initial conditions and solving for $X(s)$:

$$(s+2)X(s) = 5\frac{1}{s^2+1}$$

$$X(s) = \frac{1 + 5\left(s^2+1\right)}{\left(s^2+1\right)(s+2)} = \frac{As+B}{s^2+1} + \frac{C}{s+2}$$

Solving for the constants A, B, and C:

$$5s^2 + 6 = (As + B)(s + 2) + C(s^2 + 1)$$
$$5s^2 + 6 = (A + C)s^2 + (2A + B)s + (2B + C)$$
$$A + C = 5$$
$$2A + B = 0$$
$$2B + C = 6$$

$$A = -1/5$$
$$B = 2/5$$
$$C = 26/5$$

Thus: $$X(s) = \frac{s+2}{5(s^2+1)} + \frac{26}{5(s+2)} = -\frac{s/5}{s^2+1} + \frac{2/5}{s^2+1} + \frac{26/5}{(s+2)}$$

And, the inverse Laplace transform gives the final solution of:

$$x(t) = -\tfrac{1}{5}\cos t + \tfrac{2}{5}\sin t + \tfrac{26}{5}e^{-2t}$$

Example 3: Given the linear differential equation:

$$\ddot{x} + 5\dot{x} + 6x = 3e^{-3t}$$

with the initial conditions $\dot{x}(0) = x(0) = 1$. By Laplace transform:

$$\left[s^2 X(s) - sx(0) - \dot{x}(0)\right] + 5\left[sX(s) - x(0)\right] + 6\left[X(s)\right] = \frac{3}{s-3}$$

Substituting initial conditions and solving for X(s):

$$\left(s^2 + 5s + 6\right)X(s) = (s+6) + \frac{3}{s+3}$$

$$X(s) = \frac{(s+6)(s+3)+3}{(s+3)\left(s^2+5s+6\right)} = \frac{s^2+9s+21}{(s+2)(s+3)^2} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{(s+3)^2}$$

Solving for the constants *A*, *B*, and *C*:

$$5s^2 + 9s + = (A + B)s^2 + (6A + 5B + C)s + (9A + 6B + 2C)$$
$$A + B = 1$$
$$6A + 5B + C = 9$$
$$9A + 6B + 2C = 21$$
$$A = 7$$
$$B = -6$$
$$C = -3$$

Thus: $$X(s) = \frac{7}{s+2} + \frac{6}{s+3} + \frac{3}{(s+3)^2}$$

And, the inverse Laplace transform gives the final solution of:

$$x(t) = 7e^{-2t} - 6e^{-3t} - 3te^{-3t}$$

Example 4: Given the linear differential equation:

$$\ddot{x} + 2\dot{x} + 10x = 5t + 1$$

with quiescent initial conditions. By Laplace transform:

$$\left(s^2 + 2s + 10\right)X(s) = \frac{5}{s^2} + \frac{1}{s}$$

Solving for X(s): $$X(s) = \frac{s+5}{s^2\left(s^2+2s+10\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{\left(s^2+2s+10\right)}$$

Solving for the constant *A* and *B*:

$$s + 5 = As(s^2 + 2s + 10) + B(s^2 + 2s + 10) + (Cs + D)\, s^2$$
$$s + 5 = (A + C)s^3 + (2A + B + D)s^2 + (10A + 2B)\, s + 10B$$
$$A = 0$$
$$B = 0.5$$
$$C = 0$$

$$D = -0.5$$

Thus: $$X(s) = \frac{0.5}{s^2} - \frac{0.5}{(s+1)^2 + (3)^2}$$

And the inverse Laplace transform gives the final solution:

$$x(t) = 0.5t - 0.167e^{-t} \sin 3t$$

3.6 References

ANON, *Vol. II Flying Qualities Evaluation*, USAF Test Pilot School, Chapter 3, 1988

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Chapter 4

Matrices Algebra

Table of Contents

4.1 Introduction

This chapter on matrices is a prerequisite for the chapters on Equations of Motion, Dynamics, Linear Controls, and Flight Control Systems. The chapter deals only with applied mathematics; therefore, the theoretical scope of the subject is limited.

The text begins with sections dealing with determinants and matrices as a prerequisite to the remainder of the chapter.

4.2 Determinants

In a restricted sense, at least, the concept of a determinant is already familiar from elementary algebra. In solving systems of two or three simultaneous linear equations, it is convenient to introduce what are called determinants of the second and third order. In this chapter we will generalize these ideas to the solution of systems of three or more linear equations.

A determinant is a function which associates a variable (real, imaginary, scalar, or vector) with an array of numbers. The determinant is denoted by vertical bars on either side of a square array of numbers. Thus, if A is an $(n \times n)$ array of numbers where i designates the rows and j designates the columns, the determinant A can be written:

$$|A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

When the elements of the i^{th} row and the j^{th} column are removed from the array, the determinant of the remaining $(n - 1) \times (n - 1)$ square array is called the first-order minor of A and is denoted by M_{ij}. It is also called the minor of a_{ij}. The signed minor, with the sign determined by the sum of the row and column, is called the cofactor of a_{ij} and is denoted by:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

The value of the determinant is equal to the sum of the products of the elements of any single row or column and their respective cofactors; i.e.:

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j-1}^{n} a_{ij}A_{ij} \qquad \text{for any single } i^{th} \text{ row, or:}$$

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i-1}^{n} a_{ij}A_{ij} \qquad \text{for any single } j^{th} \text{ column.}$$

Expanding a 2×2 determinant about the first row is the easiest. The sign of the cofactor of an element can be determined quickly by observing that the sums of the subscripts alternate from even to odd when advancing across rows or down columns, meaning the signs alternate also. For example, if:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

the signs of the associated cofactors alternate as shown:

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}$$

By deleting the row and column of a_{11}, we find its cofactor is just the element a_{22} with the sign $(-1)^{2+2}$. Likewise the cofactor for a_{12} is the element a_{21} with the sign $(-1)^{2+1}$. The sum of the two products gives us the expanded value of the determinant:

$$|A| = a_{11} A_{11} + a_{12} A_{12} = a_{11} a_{22} + a_{12} (-a_{21}) = a_{11} a_{22} - a_{12} a_{21}$$

This simple example has been shown for clarity. Actual calculation of a 2 × 2 determinant is easy if we just remember it as the difference between the cross-multiplication of the elements.

To expand a 3 × 3 determinant:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If we arbitrarily choose to expand about the first row:

$$|A| = a_{11}\, A_{11} + a_{12}\, A_{12} + a_{13}\, A_{13}$$

$$a_{11}(+1)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(+1)\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which expands to give the final solution:

$$|A| = a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{12}\,(a_{21}\, a_{33} - a_{23}\, a_{31}) + a_{13}\,(a_{21}\, a_{32} - a_{22}\, a_{31})$$

The quicker method of calculating determinants is useful for the 2 × 2 determinant, but, the row or column expansion method is better for calculating values for determinants of 3 × 3 and higher. While the general rules for evaluating determinants by hand are simple, for determinants of greater size than 3 × 3, the process becomes laborious. A 5 × 5 determinant would contain 120 terms of 5 factors each. Evaluating larger determinants is an ideal task for the computer, and standard programs are available for this task. The use of determinants for solving sets of linear equations is discussed next. Determinants are also used in solving sets of linear differential equations in Chapter 3.

4.3 Matrices

An $m \times n$ matrix is a rectangular array of quantities arranged in m rows and n columns. When there is no possibility of confusion, matrices are often represented by single capital letters. More commonly, however, they are represented by displaying the quantities between brackets:

$$A = |A| = \|a_{ij}\| = [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

Note that a_{ij} refers to the element in the i^{th} row and j^{th} column of [A]. Thus, a_{23} is the element in the second row and third column. Matrices having only one column (or one row) are called column (or row) vectors. A matrix, unlike the determinant, is not assigned a "value"; it is simply an array of quantities. Matrices may be considered as single algebraic entities and combined (added, subtracted, multiplied) in a manner similar to the combination of ordinary numbers. It is necessary, however, to observe specialized algebraic rules for combining matrices. These rules are somewhat more complicated than for "ordinary" algebra. The effort required to learn the rules of matrix algebra is well justified, however, by the simplification and organization which matrices bring to problems in linear algebra.

Two matrices having the same number of rows and the same number of columns are defined as being conformable for addition and may be added by adding the corresponding elements; i.e.:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{vmatrix} = \begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{vmatrix}$$

A scalar is a single number. A matrix of any dimension may be multiplied by a scalar by multiplying each element of the matrix by the scalar. Matrix multiplication can be defined for any two matrices only when the number of columns of the first is equal to the number of rows of the second matrix.

Multiplication is not defined for other matrices. This multiplication of two matrices can be stated mathematically as:

$$[A][B] = [C]$$
$$[a_{im}][b_{mj}] = [c_{ij}]$$
$$\left[c_{ij}\right] = \sum_{k=1}^{m} a_{ik} b_{kj} \qquad (3.1)$$

The product of a pair of, 2 × 2 matrices is:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} a_{11} \cdot b_{11} + & a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{12} \\ a_{21} \cdot b_{11} + & a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{vmatrix}$$

This example points the way to an orderly multiplication process for matrices of any order. In the indicated product of equation 14.1, the left-hand factor [*A*] may be thought of as a bundle of row-vectors and the right-hand factor [*B*] may be thought of as a bundle of column-vectors. If the rows of [A] and the columns of [*B*] are treated as vectors, then cij in the resulting product [*C*] is the dot product of the i^{th} row of [*A*] and the j^{th} column of [*B*]. This rule holds for matrices of any size. Matrix multiplication is therefore a "row on column" process. The indicated product [*A*][*B*] can be carried out if [*A*] and [*B*] are conformable; again, for conformability in multiplication, the number of columns in [A] must equal to the number of rows in [*B*]. A matrix comprised of row vectors may be transformed into a matrix of column vectors by transposing rows and columns. The transpose of matrix [*A*], labeled $[A]^T$, is formed by interchanging the rows and columns of [*A*]. That is, the j^{th} row vector becomes the j^{th} column vector, and visa-versa.

Matrix algebra differs significantly from "ordinary" algebra in that multiplication is not commutative. And, because multiplication is non-commutative, care must be taken in describing the product [*C*] = [*A*][*B*] to say that [*B*] is pre-multiplied by [*A*] or, equivalently, that [*A*] is post-multiplied by [*B*].

The identity (or unit) matrix [*I*] occupies the same position in matrix algebra that the value of unity does in ordinary algebra. That is, for any matrix [*A*]:

$$[I][A] = [A][I] = [A]$$

The identity [*I*] is a square matrix consisting of ones on the principle diagonal and zeros everywhere else; i.e.:

$$[I] = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

The order (the number of rows and columns) of an identity matrix depends entirely on the requirement for conformability with adjacent matrices.

Now that matrix multiplication has just been defined; it is natural to inquire next if there is some way to divide matrices. There is not, properly speaking, a division operation in matrix algebra; however, an equivalent result is obtained through the use of the inverse matrix. In ordinary algebra, every number a (except zero) has a multiplicative inverse a^{-1} defined as follows:

$$a \cdot a^{-1} = a^{1-1} = a^0 = 1$$

In the same way, the matrix $[A]^{-1}$ is called the inverse matrix of [*A*] since:

$$[A][A]^{-1} = [A]^{-1}[A] = [A]0 = [I]$$

Matrices which cannot be inverted are called singular. For inversion to be possible, a matrix must possess a determinant not equal to zero. There is a straightforward four-step method for computing the inverse of a given matrix [*A*]:

Step 1 Compute the determinant of [*A*]. This determinant is written |*A*|. If the determinant is zero or does not exist, the matrix [A] is defined as singular and an inverse cannot be found.

Step 2 Transpose matrix [A]. The resultant matrix is written $[A]^T$.

Step 3 Replace each element a_{ij} of the transposed matrix by its cofactor A_{ij}. This resulting matrix is defined as the adjoint of matrix [A] and is written Adj[A].

Step 4 Divide the adjoint matrix by the scalar value of the determinant of [A] which was computed in Step 1. The resulting matrix is the inverse and is written $[A]^{-1}$.

From the definition of the inverse matrix, $[A]^{-1}[A] = [I]$, the computed inverse may be checked.

Consider the set of algebraic equations:

$$\begin{array}{ccccc} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n & = y_1 \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n & = y_1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_{m1}x_1 & a_{m2}x_2 & \cdots & a_{mn}x_n & = y_n \end{array} \tag{3.2}$$

That is:

$$[A]\,[X] = [Y]$$

Assuming that the inverse of [A] has been computed, both sides of this equation may be pre-multiplied by $[A]^{-1}$, giving:

$$[A]^{-1}\,[A]\,[X] = [A]^{-1}\,[Y]$$

From the definition of the inverse matrix:

$$[I]\,[X] = [A]^{-1}\,[Y]$$

we get, finally:

$$[X] = [A]^{-1}\,[Y]$$

Thus, the system of equation (3.2) may be solved for $x_1, x_2, \cdots, x_n$ by simply computing the inverse of [A]. Solution of sets of simultaneous equations using matrix algebra techniques has wide application in a variety of engineering problems and will be used extensively in this text.

Two example problems will help clarify the matrix procedures described above.

Example 1: Find $[A]^{-1}$, if

$$[A] = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 0 & 2 & -1 \end{bmatrix} \tag{3.3}$$

Step 1. Compute the determinant of [A]. Expanding about the first row

$$[A] = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$[A] = 3(-5-2) - 2(-1+0) + 0(2-1)$$

$$[A] = -21 + 2 + 0 = -19$$

The determinant has the value −19; therefore an inverse can be computed.

Step 2. Transpose [A].

$$[A]^T = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Step 3. Replace each element a_{ij} of [A] by its cofactor A_{ij} to determine the adjoint matrix. Note that signs alternate from a positive A_{11}.

$$[A]^T = \begin{bmatrix} \begin{vmatrix} 5 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} \end{bmatrix} = \begin{vmatrix} -7 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & -6 & 13 \end{vmatrix}$$

Step 4. Divide by the scalar value of determinant of [A] which was computed as –19 in step 1.

$$[A]^{-1} = \frac{1}{-19}\begin{bmatrix} -7 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & -6 & 13 \end{bmatrix}$$

Product Check

From the definition of the inverse matrix

$$[A]^{-1}\,[A] = [I]$$

This fact may be used to check a computed inverse. In the case just completed

$$[A]^{-1}[A] = \frac{1}{-19}\begin{bmatrix} -7 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & -6 & 13 \end{bmatrix}$$

$$[A]^{-1}[A] = \frac{1}{-19}\begin{bmatrix} -19 & 0 & 0 \\ 0 & -19 & 0 \\ 0 & 0 & -19 \end{bmatrix}$$

$$[A]^{-1}[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A]^{-1}[A] = [I]$$

Since the product does come out to be the identity matrix, the computation was correct.

Example 2: Given the following set of simultaneous equations, solve for x_1, x_2, and x_3.

$$\begin{aligned} 3x_1 + 2x_2 - 2x_3 &= y_1 \\ -x_1 + x_2 + 4x_3 &= y_2 \\ 2x_1 - 3x_2 + 4x_3 &= y_3 \end{aligned} \tag{3.4}$$

This set of equations can be written as:

$$[A]\;[x] = [y]$$

and solved as follows:

$$[x] = [A]^{-1}\,[y]$$

Thus, the system of equations (14.11) can be solved for the values of x_1, x_2, and $x_3$by computing the inverse of [A].

$$[A]\ [x] = [y]$$

$$\begin{bmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Step 1. Compute the determinant of [A]. Expanding about the first row

$$|A| = 3(4 + 12) - 2\ (-4\ -8) -2\ (3 - 2)$$
$$|A| = 48 + 24\ -2 = 70$$

Step 2. Transpose [A].

$$[A]^T = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & -3 \\ -2 & 4 & 4 \end{bmatrix}$$

Step 3. Determine the adjoint matrix by replacing each element in $[A]^T$ by its cofactor.

$$adj[A] = \begin{bmatrix} \begin{vmatrix} 1 & -3 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 4 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$adj[A] = \begin{vmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{vmatrix}$$

Step 4. Divide by the scalar value of the determinant of [A] which was computed as 70 in Step 1.

$$[A]^{-1} = \frac{1}{70}\begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}$$

Product Check

$$[A]^{-1}\ [A] = [I\ ]$$

$$[A]^{-1}[A]=\frac{1}{70}\begin{bmatrix}16 & -2 & 10\\ 12 & 16 & -10\\ 1 & 13 & 5\end{bmatrix}\begin{bmatrix}3 & 2 & -2\\ -1 & 1 & 4\\ 2 & -3 & 4\end{bmatrix}$$

$$[A]^{-1}[A]=\frac{1}{70}\begin{bmatrix}70 & 0 & 0\\ 0 & 70 & 0\\ 0 & 0 & 70\end{bmatrix}$$

$$[A]^{-1}[A]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Since the product in the above equation is the identity matrix, the computation is correct. The values of x_1, x_2, and x_3 can now be found for any y_1, y_2, and y_3 by pre-multiplying [y] by $[A]^{-1}$.

$$[x] = [A]^{-1}[y]$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\frac{1}{70}\begin{bmatrix}16 & -2 & 10\\ 12 & 16 & -10\\ 1 & 13 & 5\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

For example, if $y_1 = 1$, $y_2 = 13$, and $y_3 = 8$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\frac{1}{70}\begin{bmatrix}16 & -2 & 10\\ 12 & 16 & -10\\ 1 & 13 & 5\end{bmatrix}\begin{bmatrix}1\\ 13\\ 8\end{bmatrix}$$

$$x_1=\frac{1}{70}(16-26+80)=\frac{70}{70}=1$$

$$x_2=\frac{1}{70}(12+208-80)=\frac{140}{70}=2$$

$$x_3=\frac{1}{70}(1+169+40)=210=3$$

4.4 Cramer's Rule

It is often useful to have a formula for the solution of a system of equations that can be used to study properties of the solution without solving the system. Cramer's rule establishes a formula for systems of n equations in n unknowns.

If $AX = B$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1=\frac{\det(A_1)}{\det(A)},\, x_2=\frac{\det(A_2)}{\det(A)},\cdots,x_n=\frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the j the column of A by the entries in the matrix.

$$B=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$$

Proof. If $\det(A) \neq 0$, then A is invertible and $X = A^{-1}B$ is the unique solution of $AX = B$. Therefore we have

$$X = A^{-1}B = \frac{1}{\det(A)} adj(A)B = \frac{1}{\det(A)} \begin{vmatrix} C_{11} & C_{12} & \cdots & C_{n1} \\ C_{21} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{vmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Multiplying the matrices out gives

$$X = \frac{1}{\det(A)} \begin{bmatrix} b_1 & C_{11} & + & b_2 & C_{12} & + & \cdots & + & b_n & C_{n1} \\ b_1 & C_{21} & + & b_2 & C_{22} & + & \cdots & + & b_n & C_{n2} \\ \vdots & \vdots & + & \vdots & \vdots & + & & + & \vdots & \vdots \\ b_1 & C_{1n} & + & b_2 & C_{2n} & + & \cdots & + & b_n & C_{nn} \end{bmatrix}$$

The entry in the j^{th} row of X is therefore

$$X_j = \frac{b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}}{\det(A)}$$

Now let

$$A_j = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1_{j-1}} & b_1 & a_{1_{j+1}} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2_{j-1}} & b_2 & a_{2_{j+1}} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{n_{j-1}} & b_n & a_{n_{j+1}} & \cdots & a_{nn} \end{bmatrix}$$

Since A_j differs from A only in the j^{th} column, the cofactors of entries of b_1, b_2, . . . , b_n in A_j are the same as the cofactors of the corresponding entries in the j^{th} column of A. The cofactor expansion of det (A_j) along the j^{th} column is therefore

$$\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + b_n C_{nj}$$

Substituting this result gives

$$x_j = \frac{\det(A_j)}{\det(A)} \quad \text{Reference 14.3}$$

4.4.1 Example

Use Cramer's Rule to solve

$$\begin{array}{rcrcrcr} x_1 & + & & + & 2x_3 & = & 6 \\ -3x_1 & + & 4x_2 & + & 6x_3 & = & 30 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 8 \end{array}$$

4.4.2 Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}, x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

To solve a system of n equations in n unknowns by Cramer's Rule, it is necessary to evaluate determinants of $n \times n$ matrices. For systems with more than three equations, Gaussian elimination is superior computationally since it is only necessary to reduce one n by $n + 1$ augmented matrix. Cramer's rule, however gives a formula for the solution.

4.5 References

4.1 U.S.A.F. Test Pilot School, *Volume II, Flying Qualities*, Chapter 2, Vectors and Matrices, January 1988.

4.2 Shames, Irving H., *Engineering Mechanics: Statics and Dynamics, 2nd Edition*, Prentice-Hall, Inc., 1967.

4.3 Anton, Howard, *Elementary Linear Algebra*, John Wiley & Sons, 1981

Volume 1 – Math & Physics for Flight Testers

Chapter 5

Vector Algebra

Table of Contents

5.1 Vector and Scalar Algebra

Physically, a *vector* is an entity such as force, moment, velocity, or acceleration, which possesses *both magnitude and direction*. In addition to vectors, entities such as mass, inertia, volume, and work, which possess *only magnitude* are known as *scalars*. To distinguish vectors from scalars, a vector quantity is usually indicated by putting a bar above the symbol; thus $\overline{F}$, $\overline{M}$, $\overline{v}$ and $\overline{a}$ will be use to represent force, moment, velocity, and acceleration, respectively. Sometimes, boldface type is used to indicate a vector such as "***F***". The magnitude of the vector is indicated by enclosing the symbol for the vector between absolute value bars, $\left|\overline{F}\right|$. Graphically, a scalar quantity can be adequately represented by a mark on a fixed scale. Representing a vector quantity requires a directed line segment whose direction is the same as the direction of the vector and whose measured length is equal to the magnitude of the vector. The direction of a vector is described by a single angle in a two dimensional environment. Two angles are required to describe a vector in a three dimensional environment. The cosines of these angles are called direction cosines.

5.1.1 Vector Addition

The sum of two vectors $\overline{A}$ and $\overline{B}$ using the parallelogram law is illustrated in Figure 5.1a. If $\overline{A}$ and $\overline{B}$ are drawn from the same point or origin, and if the parallelogram having $\overline{A}$ and $\overline{B}$ as adjacent sides constructed, then the sum $\overline{A}+\overline{B}$ can be defined as the vector represented by the diagonal of this parallelogram which passes through the common origin of $\overline{A}$ and $\overline{B}$. Vectors can also be added by drawing them "nose-to-tail" as shown in Figures 5.1b and c.

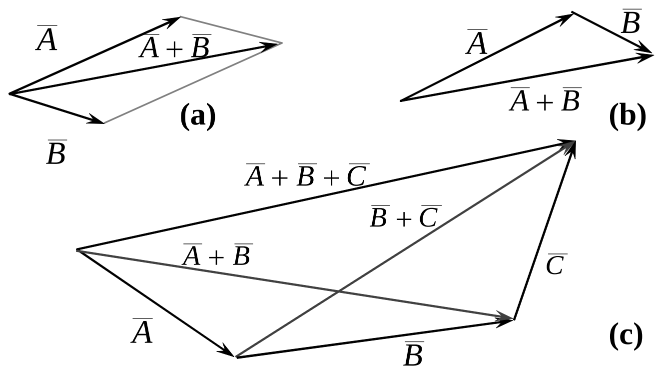

Figure 5.1 Vector Addition

5.1.2 Vector Subtraction

Vector subtraction is defined as the difference of two vectors $\overline{A}$ and $\overline{B}$:

$$\overline{A}-\overline{B}=\overline{A}+\left(-\overline{B}\right)$$

where $\left(-\overline{B}\right)$ is defined as a vector with the same magnitude but opposite direction as $\overline{B}$ (see Figure 5.2).

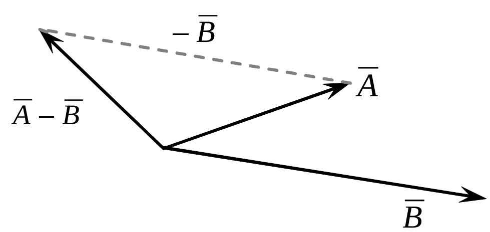

Figure 5.2 Vector Subtraction

5.1.3 Vector-Scalar Multiplication

The product of a vector and a scalar follows algebraic rules. The product of a scalar *m* and a vector $\overline{A}$ is the vector $m\overline{A}$, whose length is the product of the absolute value of *m* and the magnitude of $\overline{A}$, and whose direction is the same as the direction of $\overline{A}$ if *m* is positive, and opposite to it if *m* is negative.
If $\overline{A}$, $\overline{B}$, and $\overline{C}$ are vectors and m and n are scalars, then the following mathematical rules apply:

$m\overline{A}=\overline{A}m$	Commutative
$m\left(n\overline{A}\right)=\left(mn\right)\overline{A}$	Associative

$$(m+n)\overline{A} = m\overline{A} + n\overline{A} \qquad \text{Distributive}$$

$$m(\overline{A} + \overline{B}) = m\overline{A} + m\overline{B} \qquad \text{Distributive}$$

These laws involve multiplication of a vector by one or more scalars. Products of vectors will be defined later.

5.1.4 Vector Components

Any vector in three-dimensional space can be represented with the initial point at the origin of a rectangular coordinate system as shown in Figure 5.3. The vector from the origin to a point in the coordinate system is called a position vector, so the vector $\overline{A}$ in Figure 5.3 is the position vector for point *P*. The perpendicular projection of the vector on each of the axes gives the component of the vector along that axis. Note that the vector sum of the components graphically gives the magnitude and direction of the original vector as a result.

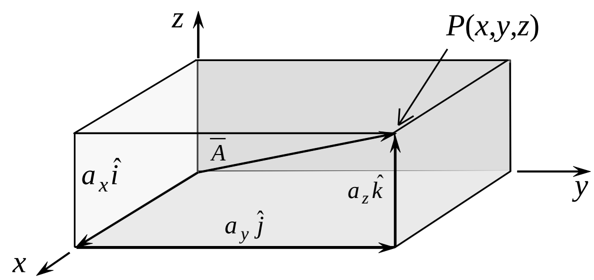

Figure 5.3 Components of a Vector

Regardless of its direction, a vector whose length is one (unity) is called a unit vector. If $\hat{a}$ is defined as $\frac{\overline{A}}{|\overline{A}|}$, then $\hat{a}$ is a unit vector having the same direction as $\overline{A}$ and a magnitude of one. The components of $\hat{a}$ are the cosines of the angles necessary to define the direction. Components of a unit vector in a rectangular coordinate system are usually designated by $\hat{i}$, $\hat{j}$, and $\hat{k}$ with a carat symbol over them. In Figure 5.3, the components of the vector $\overline{A}$ are magnitudes a_x, a_y, and a_z along the *x*, *y*, and *z* axes, respectively. The sum or resultant of the components can be expressed as:

$$\overline{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

And the magnitude of *A* is easily calculated as: $|\overline{A}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

An arbitrary vector from initial point *P* to terminal point *Q* such as shown in Figure 5.4 can be represented in terms of unit vectors, also.

We can first write the position vectors for the two points *P* and *Q*:

$$\overline{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \qquad \overline{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

Then using vector addition:

$$\overline{r}_1 + \overline{PQ} = \overline{r}_2$$

or: $\overline{PQ} = \overline{r}_1 - \overline{r}_2 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

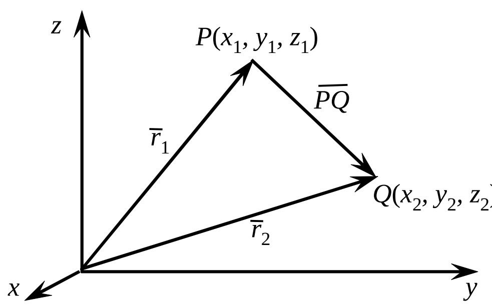

Figure 5.4 Arbitrary Vector Representation

5.1.5 Vector Dot Product

In addition to the product of a scalar and a vector, two other types of products are defined in vector analysis. The first of these is the dot product (or scalar product), denoted by a dot between the two vectors. The dot product is an operation between two vectors resulting in a scalar quantity (thus the name scalar product). Analytically, it is calculated by adding the products of like components. That is, if:

$$\overline{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \qquad \overline{B} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$

then:

$$\overline{A} \cdot \overline{B} = a_x b_x + a_y b_y + a_z b_z$$

Geometrically, it is equal to the product of the magnitudes of the two vectors and the cosine of the angle between them (the angle is measured in the plane formed by the two vectors, if they had the same origin). The dot product can be written:

$$\overline{A} \cdot \overline{B} = \left|\overline{A}\right|\left|\overline{B}\right| cos\,\theta$$

Some interesting applications of the dot product are the geometric implications. For instance, the geometric, scalar projection of one vector onto another is shown in Figure 5.5. Using trigonometry, the projection of $\overline{A}$ on $\overline{B}$ is seen to be equal to $\left|\overline{A}\right|\cos\theta$. A quick method to calculate such a projection without knowing the angle is to calculate the dot product and divide by the magnitude of the vector being projected onto. That is, the projection of $\overline{A}$ onto $\overline{B}$ is equal to $\frac{\overline{A}\cdot\overline{B}}{\left|\overline{B}\right|} = \left|\overline{A}\right|\cos\theta$.

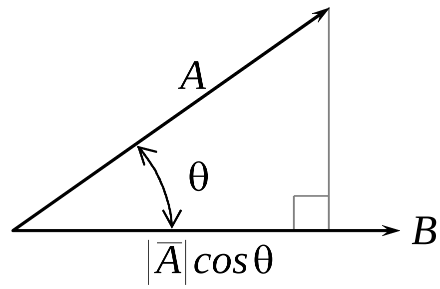

Figure 5.5 Geometric Projection of Vectors

Several particular dot products are worth mentioning. If one of the vectors is a unit vector, for example, the dot product becomes:

$$\hat{i} \cdot \overline{B} = \left|\hat{i}\right|\left|\overline{B}\right|\cos\theta = (1)\left|\overline{B}\right|\cos\theta = \left|\overline{B}\right|\cos\theta$$

which is the projection of $\overline{B}$ on $\hat{i}$ or more importantly the component of $\overline{B}$ in the direction of $\hat{i}$. Also note that the dot product of a vector with itself is just equal to the magnitude squared, since the angle is zero and cos θ = 1. More useful is the situation where two vectors are perpendicular (orthogonal). The dot product is zero because the cosine of 90 degrees is zero. Thus, the dot product may be a test of orthogonality. Examples of these properties using standard unit vectors are:

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \qquad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

If $\overline{A}$, $\overline{B}$, and $\overline{C}$ are vectors and *m* is a scalar, then the following mathematical rules apply:

$\overline{A} \cdot \overline{B} = \overline{B} \cdot \overline{A}$	Commutative
$\overline{A} \cdot \left(\overline{B} + \overline{C}\right) = \overline{A} \cdot \overline{B} + \overline{A} \cdot \overline{C}$	Distributive
$m\left(\overline{A} \cdot \overline{B}\right) = \left(m\overline{A}\right) \cdot \overline{B} = \overline{A} \cdot \left(m\overline{B}\right)$	Associative

5.1.6 Vector Cross Product

The third type of product involving vector operations is the cross product (or vector product) denoted by placing a "cross" between two vectors. By definition, the cross product is an operation between two vectors which results in another vector (thus, the name vector product). Analytically, the cross product is calculated for three-dimensional vectors by a top row expansion of a determinant:

$$\overline{A}\times\overline{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\hat{i}+(-1)\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\hat{j}+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\hat{k}$$

$$=\left(a_y b_z - a_z b_y\right)\hat{i}+\left(a_z b_x - a_x b_z\right)\hat{j}+\left(a_x b_y - a_y b_x\right)\hat{k}$$

The geometric definition has to be approached carefully because it does not imply a vector. The magnitude (scalar) of the cross product is equal to the product of the two magnitudes and the sine of the angle between the two vectors:

$$\overline{A}\times\overline{B}=\left|\overline{A}\right|\left|\overline{B}\right|\sin\theta$$

While the magnitude is determined as shown above, the direction of the resultant cross product vector is always orthogonal to the plane formed by the crossed vectors. The sense is such that when the fingers of the right hand are curled from the first vector to the second, the thumb points in the direction of the cross product as shown in Figure 5.6. Note the importance of the order in writing $\overline{A}\times\overline{B}$ since $\overline{A}\times\overline{B}\neq\overline{B}\times\overline{A}$.

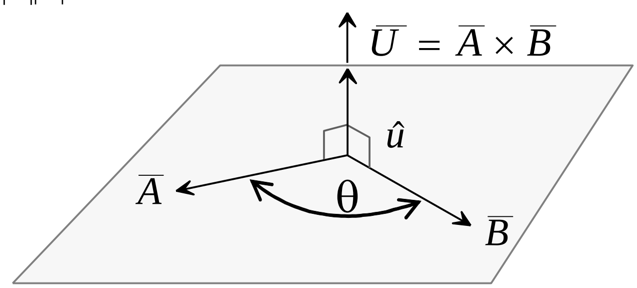

Figure 5.6 Geometric Definition of the Cross Product

Some practical applications of the above definitions using sine of zero and 90 degrees are:

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$$

$$\hat{i}\times\hat{j}=\hat{k}\qquad \hat{j}\times\hat{k}=\hat{i}\qquad \hat{k}\times\hat{i}=\hat{j}$$

$$\hat{j}\times\hat{i}=-\hat{k}\qquad \hat{k}\times\hat{j}=-\hat{i}\qquad \hat{i}\times\hat{k}i=-\hat{j}$$

5.1.7 Vector Differentiation

For a *scalar function*, we are concerned only with the variation in magnitude of some quantity which is changing with time. The definition of the time derivative of a scalar function of the variable *t* is defined as:

$$\frac{df(t)}{dt}=\lim_{\Delta t\to 0}\left[\frac{f(t+\Delta t)-f(t)}{\Delta t}\right]$$

In the case of a *vector function*, however, the variation in time may be a change in *magnitude* or it may be a change in *direction* or both. The time derivative of a vector function with respect to time has the same form as the equation above:

$$\frac{d\overline{F}(t)}{dt}=\lim_{\Delta t\to 0}\left[\frac{\overline{F}(t+\Delta t)-\overline{F}(t)}{\Delta t}\right]$$

If $\overline{F}$ has no change in direction during the time interval, this operation differs little from the scalar case. However, when this is not the situation, we find, for the derivative of $\overline{F}$, a new vector having a magnitude as well as a direction that is different from $\overline{F}$ itself.

Consider the rate of change of the position vector of a particle with respect to time. Following the definition given by the previous equation, we have:

$$\frac{d\overline{r}}{dt}=\lim_{\Delta t\to 0}\left[\frac{\overline{r}(t+\Delta t)-\overline{r}(t)}{\Delta t}\right]$$

The position vectors given in the brackets are shown in Figure 5.7.

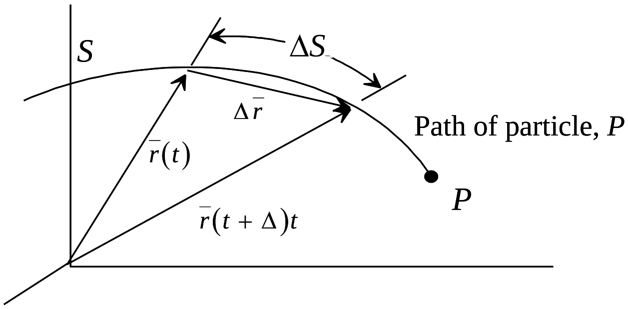

Figure 5.7 The Derivative of a Position Vector

The difference between position vectors $\bar{r}(t+\Delta t)$ and $\bar{r}(t)$ gives rise to the vector $\Delta\bar{r}$, which is shown as a chord connecting two points along the path of the particle. The arc length of the trajectory for some Δt is Δs. The derivative for the vector $\bar{r}(t)$ can be expanded by multiplying by the quantity $\frac{\Delta s}{\Delta s}=1$, as follows:

As $\Delta t \rightarrow 0$, , and its direction becomes tangential to the trajectory. The $\Delta s/\Delta t$ portion gives the magnitude of the derivative which should be noted as the speed of the particle or the change in the position per unit time. In the limit, $\Delta\bar{r}/\Delta s$ becomes a unit vector $\hat{\varepsilon}_t$ tangent to the trajectory. In, summary, the first derivative of a position vector is a vector tangential to the trajectory with a magnitude equal to the speed of the particle.

Using a different approach, in taking the derivative of a vector written in the form of magnitude times a unit vector, we get:

$$\frac{d\bar{r}(t)}{dt}=\frac{d\left[r(t)\hat{r}\right]}{dt}=\frac{dr(t)}{dt}\hat{r}+r(t)\frac{d\hat{r}}{dt}$$

Note that the linear velocity using this form of a vector has two components, the first is the rate of change of the scalar function with a direction the same as the original vector itself. The second component is the scalar function itself with the rate of change of the unit vector as its direction. We know that the unit vector doesn't change magnitude, but, it may change direction giving a non-zero derivative.

Now consider the differentiation of vectors undergoing certain algebraic operations. For vector functions $\bar{A}$ (t) and $\bar{B}$ (t), and scalar function $f(t)$, the following mathematical rules apply:

$$\frac{d\left(\bar{A}+\bar{B}\right)}{dt}=\frac{d\bar{A}}{dt}+\frac{d\bar{B}}{dt}$$ Distributive Derivative

$$\frac{d\left(\bar{A}\cdot\bar{B}\right)}{dt}=\bar{A}\cdot\frac{d\bar{B}}{dt}+\frac{d\bar{A}}{dt}\cdot\bar{B}$$ Dot Product Derivative

$$\frac{d\left(\bar{A}\times\bar{B}\right)}{dt}=\bar{A}\times\frac{d\bar{B}}{dt}+\frac{d\bar{A}}{dt}\times\bar{B}$$ Cross Product Derivative

$$\frac{d}{dt}\left[f(t)\bar{B}\right]=f(t)\frac{d\bar{B}}{dt}+\frac{df(t)}{dt}\bar{B}$$ Scalar-Vector Product Derivative

5.2 Kinematics

The time derivative of a position vector relative to some reference system is the *linear velocity*. Note in particular that the velocity of a particle is a vector that has direction and magnitude. The magnitude of the velocity is referred to as speed. The derivative of a velocity vector or the second derivative of a position vector is the *linear acceleration*.

Linear velocity and acceleration have meaning only if expressed in reference to another point and only if relative to a particular frame of reference. In this text, for discussions of single reference systems, the linear velocity and acceleration will always be relative to the origin of the reference frame in which the problem is given and will be denoted by single letters, e.g., $\bar{V}$ and $\bar{a}$. If there are two reference systems in the problem, the notation will be changed to include the reference frame as a subscript, i.e., $\bar{V}_A$

or $\overline{V}_B$. To take a time derivative of vector relative to a particular reference system, the notation will be, $\left.\frac{d\overline{F}}{dt}\right|_A$ or $\left.\frac{d\overline{F}}{dt}\right|_B$. By introducing the concept of multiple reference systems, it is appropriate to discuss the chain rule. For two reference systems, the chain rule is simply stated as follows: for point *P* in reference system *B* (which in turn is in another reference system *A*), the velocity of *P* relative to reference system *A* is equal to:

$$\overline{V}_{P/A} = \overline{V}_{P/B} + \overline{V}_{B/A}$$

Another method of determining velocities and accelerations will be determined using pure translation and rotation. Simplification will consist of very specific problems with convenient alignment of reference systems at specific instances in time. It will appear that the time element has disappeared in the following analysis since the vectors will be constant at the instant we observe them. The two basic motions, translation and rotation, will be applied to a rigid body which is assumed not to bend or twist (every point in the body remains equidistant from all other points). It will become important to determine not only the velocity and acceleration of a point in a rigid body, but also that of a vector which lies in the body.

5.2.1 Translation

If a body moves so that all particles have the same velocity relative to some reference at any instant of time, the body is said to be in pure translation. A vector in pure *translation* changes neither its magnitude or direction while translating, so its first derivative is zero. An example would be a vector from the center of gravity to the wingtip of an airplane in straight and level, unaccelerated flight with respect to a reference system attached to the earth's surface. From the ground, the vector never changes magnitude or direction, although every point on the aircraft is traveling at the same velocity (see Figure 5.8a).

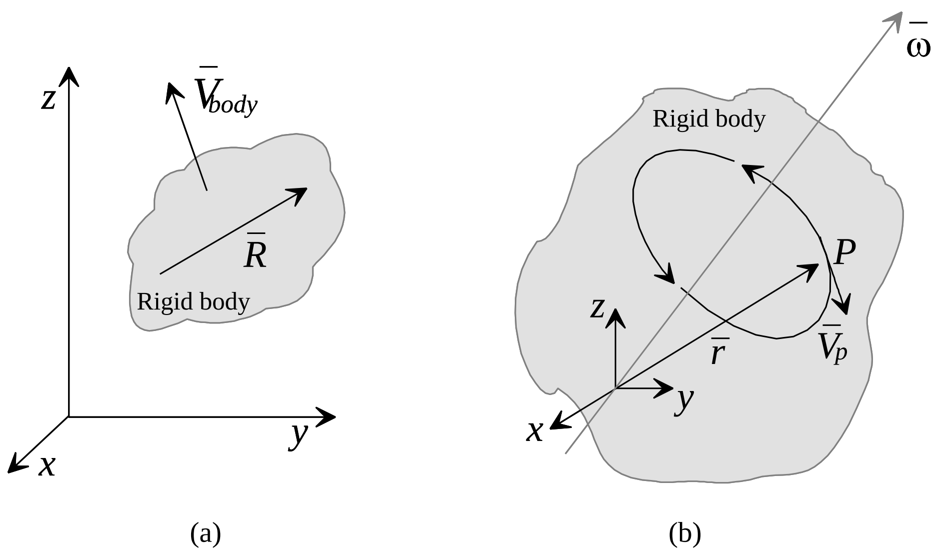

Figure 5.8 Translation and Rotation of Vectors in Rigid Bodies

5.2.2 Rotation

If a body moves so that the particles along some line in the body have a zero velocity relative to some reference, the body is said to be in pure rotation relative to this reference. The line of stationary particles shown in Figure 5.8b is called the axis of rotation. A free vector that describes the rotation is called the angular velocity, $\overline{\omega}$, and has direction determined by the axis of rotation, using the right-hand rule to determine the sense. The chain rule as described for linear velocity also applies to the angular velocity, as does a definition of its magnitude being rotation speed. The first derivative of angular velocity is the angular acceleration.

5.2.3 Combination of Translation and Rotation in One Reference System

Two simple motions of a body have been considered, namely, translation and rotation. It will now demonstrated that at each instant, the motion of any rigid body can be thought of as the superposition of both a translational and a rotational motion. Consider, for simplicity, a body moving in a plane with the positions of the body at times t and ($t + \Delta t$) as shown in Figure 5.9. Select any point B in the body. Imagine that the body is displaced without rotation from its position at time t to the position at time ($t + \Delta t$) so that point B reaches its correct final position B'. The displacement vector for this translation is shown as . To reach the correct orientation at ($t + \Delta t$), the body must be rotated through an angle $\Delta\bar{\varphi}$ about an axis of rotation normal to the plane and passing through point B'. This rotation now places point C at its correct final position C'.

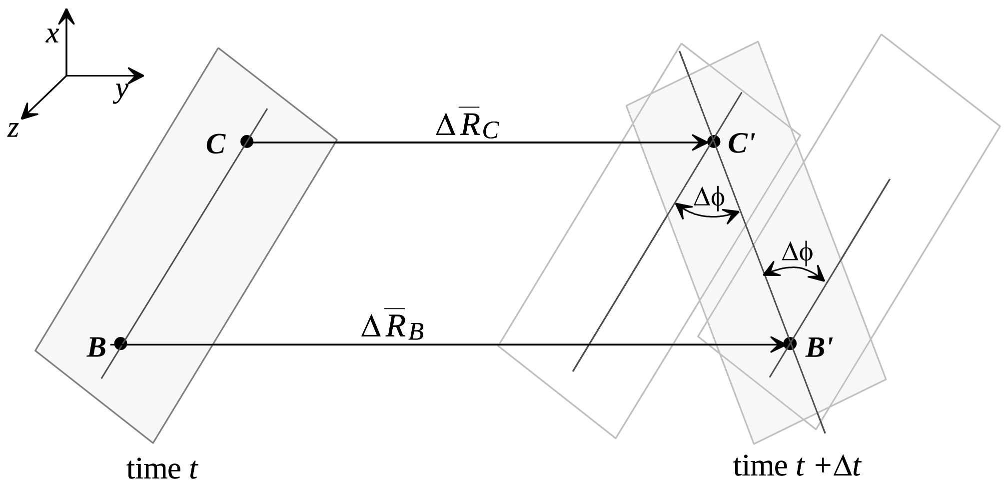

Figure 5.9 Superposition of Translational and Rotational Motions

Consider now the ratios $\frac{\Delta\bar{R}}{\Delta t}$ and $\frac{\Delta\bar{\varphi}}{\Delta t}$. These may be taken as average translational and rotational velocities, respectively, of the body, which we could superimpose to get from the initial position to the final position in the time Δt. In the limit by letting $\Delta t \rightarrow 0$, there is instantaneous translational and rotational velocities which, when superimposed, give the instantaneous motion of the body.

The rigid body in Figure 5.9 has a pure angular velocity $\bar{\varpi}$ and a pure translation $\bar{V}$ in the given reference frame. The instantaneous velocity of point C is just the sum:

$$\bar{V}_C = \bar{V}_{trans} + \bar{V}_{rot} = \bar{V}_B + \bar{\varpi} \times \bar{r} \tag{5.5}$$

where r is the position vector locating the point C with reference to point B. The acceleration of point C may be calculated by taking the derivative of the velocity shown in equation 5.5:

$$\bar{a}_C = \frac{d\bar{V}_C}{dt} = \frac{d\bar{V}_B}{dt} + \frac{d(\bar{\varpi} \times \bar{r})}{dt} = \bar{a}_B + \bar{\varpi} \times \dot{\bar{r}} + \dot{\bar{\varpi}} \times \bar{r} = \bar{a}_B + \bar{\varpi} \times (\bar{\varpi} \times \bar{r}) + \dot{\bar{\varpi}} \times \bar{r} \tag{5.6}$$

In the above equation, since the distance between points B and C is fixed, the term $\dot{\bar{r}}$ has only one non-zero component due to rotation only, thus, $\dot{\bar{r}} = \bar{\varpi} \times \bar{r}$. If the translational and angular velocities are constant as they were originally assumed in Figure 5.9, the first and last terms are zero, but, it is interesting to note that one acceleration term remains even though $\bar{V}$ and $\bar{\varpi}$ were constant. This acceleration term $[\bar{\varpi} \times (\bar{\varpi} \times \bar{r})]$ is known as the *centripetal acceleration*.

5.2.4 Velocity Vector Derivatives in Different Reference Systems

The more general problem of relative motion between a point and a reference system that is itself moving relative to another reference system will now be approached. More than one reference system is often used in order to simplify the analysis of problems in rigid body dynamics. As a first step, it is necessary to examine the procedure of differentiation with respect to time in the presence of two reference systems moving relative to each other.

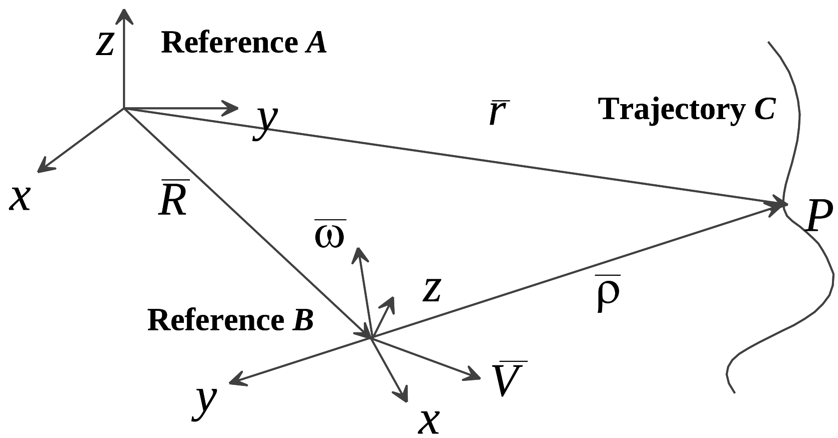

Figure 5.10 Motion with Two Reference Systems

For this purpose consider Figure 5.10, where a particle *P* is shown moving along trajectory *C*. Such a situation might be simulated by a component that is moving along a path inside an airplane (moving reference system *B*), while the airplane has a known motion relative to the ground (fixed reference *A*), given in terms of the vectors $\dot{\bar{R}}$ and $\bar{\omega}$. To reach the desired results effectively, it is helpful to express the vector $\bar{\rho}$ in terms of components parallel to the *xyz* axes of reference system B:

$$\bar{\rho} = x\hat{i} + y\hat{j} + z\hat{k}$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors for reference system *B*. Differentiating the above equation with respect to time for reference *B*, we get:

$$\left(\frac{d\bar{\rho}}{dt}\right)_B = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \tag{5.7}$$

To take the derivative of $\bar{\rho}$ with respect to time for reference *A*, it must be remembered that $\hat{i}$, $\hat{j}$ and $\hat{k}$ are moving relative to reference *A*, thus they are functions of time:

$$\left(\frac{d\bar{\rho}}{dt}\right)_A = \left(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\right) + \left(x\dot{\hat{i}} + y\dot{\hat{j}} + z\dot{\hat{k}}\right) \tag{5.8}$$

The unit vector $\hat{i}$ is a vector fixed in reference *B* and accordingly $\dot{\hat{i}}$ equals $\bar{\omega} \times \hat{i}$ for the same reasons stated above for equation 5.6. The same conclusions apply to $\hat{j}$ and $\hat{k}$. The last expression in parentheses in equation 5.8 can then be expanded to:

$$
\begin{aligned}
x\dot{\hat{i}} + y\dot{\hat{j}} + z\dot{\hat{k}} &= x\left(\overline{\omega}\times\hat{i}\right) + y\left(\overline{\omega}\times\hat{j}\right) + z\left(\overline{\omega}\times\hat{k}\right) \\
&= \left(\overline{\omega}\times x\hat{i}\right) + \left(\overline{\omega}\times y\hat{j}\right) + \left(\overline{\omega}\times z\hat{k}\right) \\
&= \overline{\omega}\times\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \overline{\omega}\times\overline{\rho}
\end{aligned}
$$

The first expression in the right-hand side of equation 5.8 can also be replaced by making a substitution (from equation 5.7). This results in:

$$\left(\frac{d\overline{\rho}}{dt}\right)_A = \left(\frac{d\overline{\rho}}{dt}\right)_B + \overline{\omega}\times\overline{\rho} \tag{5.9}$$

where $\overline{\omega}$ is the angular velocity of reference *B* relative to reference *A*.

The velocity of a particle relative to a particular reference system is the derivative, as seen from that reference of the position vector of the particle. In Figure 5.10, the velocities of the particle *P* relative to the *A* and to the *B* references are, respectively:

$$\overline{V}_A = \left(\frac{d\overline{r}}{dt}\right)_A \qquad \overline{V}_B = \left(\frac{d\overline{r}}{dt}\right)_B$$

Now it would be of interest to relate these velocities, which can be readily done by first noting that:

$$\overline{r} = \overline{R} + \overline{\rho}$$

Differentiating with respect to time for the *A* reference, we get:

$$\left(\frac{d\overline{r}}{dt}\right)_A = \overline{V}_A = \left(\frac{d\overline{R}}{dt}\right)_A + \left(\frac{d\overline{\rho}}{dt}\right)_A$$

The term $\left(\frac{d\overline{R}}{dt}\right)_A$ is the velocity of the origin of the *B* reference relative to the *A* reference, $\left(\dot{\overline{R}}\right)$. The term $\left(\frac{d\overline{\rho}}{dt}\right)_A$ can be replaced with equation (5.9). Denoting $\left(\frac{d\overline{\rho}}{dt}\right)_B$ simply as V_B, the above expression then becomes the desired relation:

$$\overline{V}_A = \overline{V}_B + \dot{\overline{R}} + \overline{\omega}\times\overline{\rho} \tag{5.10}$$

$\dot{\overline{R}} + \overline{\omega}\times\overline{\rho}$ is the "Transport Velocity" and is the only velocity if point is rigidly attached to the *B* axis system.

5.2.5 Acceleration Vector Derivatives in Different Reference Systems

The acceleration of a particle relative to a particular reference system is simply the time derivative of the velocity vector relative to the reference.

$$\overline{a}_A = \left(\frac{d\overline{V}_A}{dt}\right)_A = \left(\frac{d^2\overline{r}}{dt^2}\right)_A$$

$$\overline{a}_B = \left(\frac{d\overline{V}_B}{dt}\right)_B = \left(\frac{d^2\overline{\rho}}{dt^2}\right)_B$$

The acceleration vectors can be related for two reference systems moving arbitrarily relative to each other by differentiating the terms of equation (14.10) with respect to time for the A reference. Thus:

$$\overline{a}_A = \left(\frac{d\overline{V}_A}{dt}\right)_A = \left(\frac{d\overline{V}_B}{dt}\right)_A + \ddot{\overline{R}} + \left[\frac{d}{dt}\left(\overline{\omega}\times\overline{\rho}\right)\right]_A$$

It is convenient to carry out the derivative of the cross product using the product rule. Thus the above equation becomes:

$$\bar{a}_A = \left(\frac{d\bar{V}_B}{dt}\right)_A + \ddot{\bar{R}} + \bar{\omega} \times \left(\frac{d\bar{\rho}}{dt}\right)_A + \left(\frac{d\bar{\omega}}{dt}\right)_A \times \bar{\rho} \tag{5.11}$$

To introduce more meaningful terms, the following relations can be used:

$$\left(\frac{d\bar{\rho}}{dt}\right)_A = \left(\frac{d\bar{\rho}}{dt}\right)_B + \bar{\omega} \times \bar{\rho} \tag{5.9}$$

and similarly $$\left(\frac{d\bar{V}_B}{dt}\right)_A = \left(\frac{d\bar{V}_B}{dt}\right)_B + \bar{\omega} \times \bar{V}_B$$

Substituting into equation (5.11), results in:

$$\bar{a}_A = \left(\frac{d\bar{V}_B}{dt}\right)_B + \ddot{\bar{R}} + \bar{\omega} \times \bar{V}_B + + \bar{\omega} \times \left(\frac{d\bar{\rho}}{dt}\right)_B + \bar{\omega} \times (\bar{\omega} \times \bar{\rho}) + \left(\frac{d\bar{\omega}}{dt}\right)_A \times \bar{\rho}$$

Noting that $\left(\frac{d\bar{V}_B}{dt}\right)_B$ is $\bar{a}_B$; $\left(\frac{d\bar{\rho}}{dt}\right)_B$ is $\bar{V}_B$; and $\left(\frac{d\bar{\omega}}{dt}\right)_A$ is $\dot{\bar{\omega}}$, rearranging terms gives:

$$\bar{a}_A = \bar{a}_B + \ddot{\bar{R}} + 2(\bar{\omega} \times \bar{V}_B) + \dot{\bar{\omega}} \times \bar{\rho} + \bar{\omega} \times (\bar{\omega} \times \bar{\rho}) \tag{5.12}$$

where $\bar{\omega}$ and $\dot{\bar{\omega}}$ are the angular velocity and acceleration, respectively, of the B reference relative to the A reference. The vector $2(\bar{\omega} \times \bar{V}_B)$ is called the *Coriolis acceleration* vector.

$\ddot{\bar{R}} + \dot{\bar{\omega}} \times \bar{\rho} + \bar{\omega} \times (\bar{\omega} \times \bar{\rho})$ is the transport acceleration and is the only acceleration if point ρ is rigidly attached to the *B* axis system.

5.2.6 Velocity and Acceleration in a Single Reference System

Finally, consider a rigid body having an arbitrary motion which is expressed as some translational velocity *V* plus a rotational velocity $\bar{\omega}$ relative to reference *A* as shown in Figure 14.11. A displacement vector $\bar{\rho}$ connects point *p* in the body as shown. If a reference *B* were attached to move with the rigid body, then that $\bar{\rho}$ is a vector fixed in reference *B*. Since only one reference will be used, dots can be used alone to indicate time derivatives with no danger of ambiguity. Thus:

$$\dot{\bar{\rho}} = \bar{\omega} \times \bar{\rho}$$
$$\ddot{\bar{\rho}} = \bar{\omega} \times (\bar{\omega} \times \bar{\rho}) + \dot{\bar{\omega}} \times \bar{\rho} \tag{5.13}$$

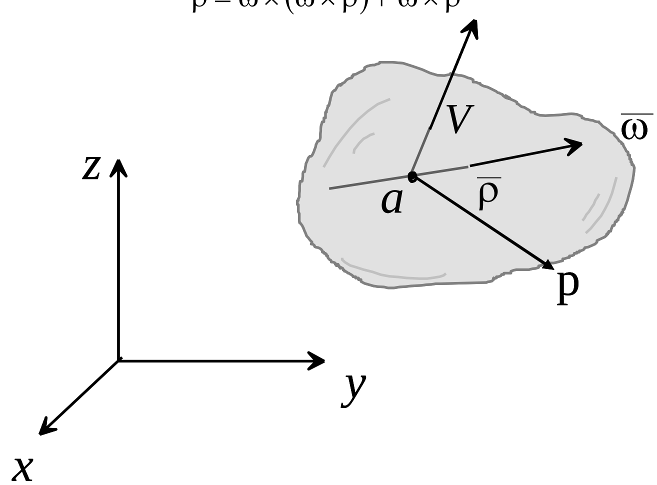

Figure 5.11 Motion of a Rigid Body

Since $\bar{\rho}$ is the position vector referencing point ρ with respect to point a, then $\dot{\bar{\rho}}$ is the velocity of point r relative to point *a* as seen from reference *A*. Furthermore, $\dot{\bar{\rho}}$ does not depend on the line of action of $\bar{\omega}$. The relative velocity between two particles, as seen from a particular reference then is actually the difference between the respective velocities of the particles as seen from that reference. Thus:

$$\dot{\bar{\rho}} = \bar{V}_p - \bar{V}_a$$

Hence, employing equation 5.13, the following equation can be formed:

$$\bar{V}_p = \bar{V}_a + \bar{\omega} \times \bar{\rho} \tag{5.14}$$

This states that the velocity of particle ρ of a rigid body as seen from reference *A* equals the velocity of any other particle a (such as velocity body's cg) of this body as seen from reference *A* plus the velocity of particle ρ relative to particle *a*. Differentiating equation 5.14 again, a relation involving the acceleration vectors of two points on a rigid body can be obtained:

$$\bar{a}_p = \bar{a}_a + \bar{\omega} \times (\bar{\omega} \times \bar{\rho}) + \dot{\bar{\omega}} \times \bar{\rho} \tag{5.15}$$

With equations 5.14 and 5.15, relations between the motions of two points of a rigid body as seen from a single reference have been formulated. Such relations will be used extensively in the study of rigid body dynamics. An example application is using inertially references GPS instrumentation to calculate total velocities and accelerations of some point on an aircraft. These concepts are difficult to realize until a few problems are attempted. For example:

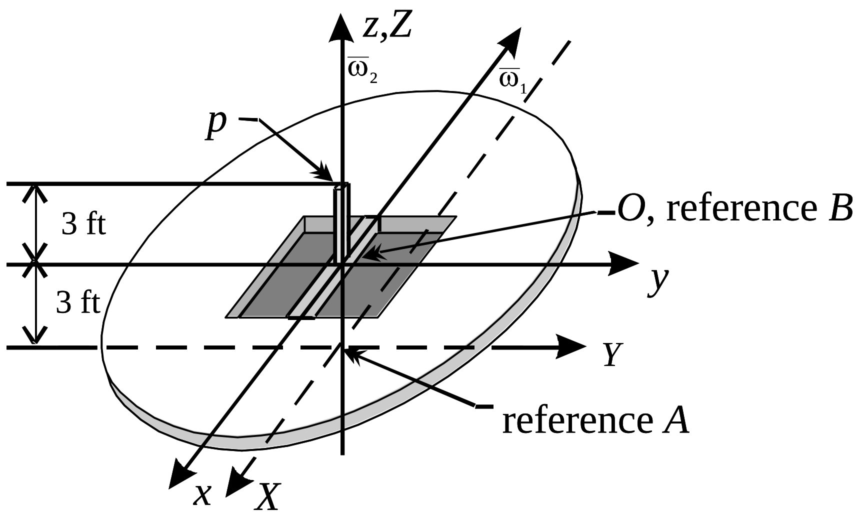

Figure 5.12 Two Reference System Problem

The angular velocity of the arm $\overline{Op}$ relative to the disk in Figure 5.12 is 10 rad/sec, shown vectorally in the diagram as $\bar{\omega}_1$, while the angular velocity of the disk relative to the ground is 5 rad/sec, shown vectorally as $\bar{\omega}_2$. The angular accelerations are zero. Reference *B* is attached to the platform, while frame *A* is fixed to the ground, three feet below the disk. At the instant in question, the arm $\overline{Op}$ is in the vertical position, and the reference axes directions coincide, although displaced.

Find the velocity and acceleration of point p relative to the fixed reference frame A. Using equation 5.10.

$$\bar{V}_{p/A} = \bar{V}_{p/B} + \bar{V}_{B/A} + \bar{\omega}_{B/A} \times \bar{r}_{p/B} \tag{5.16}$$

the second term, $\bar{V}_{B/A} = 0$, since the *B* frame is only rotating relative to *A*.

$$\bar{\omega}_{B/A} = \bar{\omega}_2 = 5\hat{k}\,\tfrac{rad}{\sec} \text{ and } \bar{r}_{p/B} = 3\hat{k} \text{ feet, from the figure.}$$

This leaves $\bar{V}_{p/B}$ which involves angular velocity $\bar{\omega}_1 = -10\hat{i}$, relative to *B*.

$$\bar{V}_{p/B} = \bar{\omega}_1 \times \bar{r}_{p/B} = \left(-10\hat{i} \times 3\hat{k}\right) = (-30)\left(-\hat{j}\right) = 30\hat{j} \text{ ft/sec}$$

Substituting all the parts into equation 5.16:

$$\bar{V}_{p/A} = 30\hat{j} + 5\hat{k} \times 3\hat{k} + 0 = 30\hat{j} + 15\left(\hat{k} \times \hat{k}\right) = 30\hat{j} \text{ ft/sec}$$

For the acceleration, the general expression is equation 5.12.

$$\bar{a}_{p/A} = \bar{a}_{p/B} + \bar{a}_{A/B} + \dot{\bar{\omega}}_{B/A} \times \bar{r}_{p/B} + 2\left(\bar{\omega}_{B/A} \times \bar{V}_{p/B}\right) + \bar{\omega}_{B/A} \times \left(\bar{\omega}_{B/A} \times \bar{r}_{p/B}\right) \tag{5.17}$$

Note that $\bar{a}_{A/B} = 0$. $\bar{a}_{p/B}$ is a centripetal acceleration due to the rotation of the arm. The centripetal acceleration may be arrived at in several different ways.

$$\begin{aligned}\bar{a}_{p/B} &= \frac{d}{dt}\bar{V}_{p/B} = \frac{d}{dt}\left(\bar{\omega}_1 \times \bar{r}_{p/B}\right) = \dot{\bar{\omega}} \times \bar{r}_{p/B} + \bar{\omega}_1 \times \dot{\bar{r}}_{p/B} \\ &= \bar{0} + \bar{\omega}_1 \times \left(\bar{\omega}_1 \times \bar{r}_{p/B}\right) = \left(-10\hat{i}\right) \times \left(30\hat{j}\right) = -300\hat{k} \text{ ft/sec}\end{aligned}$$

Substituting this value and the others already calculated into equation 5.17:

$$\bar{a}_{p/B} = -300\hat{k} + \bar{0} + \bar{0} \times 3\hat{k} + 2\left(5\hat{k} \times 30\hat{j}\right) + 5\hat{k} \times \left(5\hat{k} \times 3\hat{k}\right) = -300\bar{i} - 300\hat{k}$$

While working problems where there is a choice of axes, care must be exercised in choosing initial conditions so that as many parameters as possible are equal to zero, and most importantly so that the axes are aligned at the instant in question. Also, whether a reference system is fixed in a body or not will have profound effects on the velocities as seen from that origin. It helps immensely to visualize the velocity and acceleration from the origin to avoid confusion. Also, answers should be checked to see if they are logical, both in magnitude and direction. The right-hand rule is essential.

When working with large systems, with many variables it becomes necessary to develop a shorthand method of writing systems of equations. The use of matrix algebra is the solution.

5.3 References

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Volume 1 – Math & Physics for Flight Testers

Chapter 6

Statistics and Data Analysis

Table of Contents

Math & Physics for Flight Testers, Volume I
Chapter 6
Statistics and Data Analysis

6.1 General Introduction

Flight testing consists almost entirely of experimental observations from which we record numbers: time to climb, fuel flow, short period frequency, Cooper-Harper ratings, INS drift rate, to name a few. All experimental observations have inaccuracies. Understanding the extent of these errors and developing methods to reduce their magnitude to an acceptable level is the subject of this course.

6.1.1 Types of Errors

In discussing the errors in our experimental observations, we need to make a distinction between two very different kinds of errors: systemic errors and random errors.

Systemic errors are repeatable errors caused by some flaw in our measuring system. For example, if we measure lengths with a ruler that has the first inch broken off, our data will all have a one inch systemic error. The instrument corrections we apply to indicated airspeed and altitude to obtain calibrated airspeed and altitude is an example of compensating for a known systemic error.

Random errors are not repeatable. If we make multiple observations of the same parameter with the same equipment under the same conditions, we will still have small variations in the results. These variations are caused by unobserved changes in the experimental situation. They can result from small errors in the judgment of the observer, such as in interpolating between the marks of the smallest scale division of an instrument. Other error sources could be unpredictable variations in temperature, voltage, or friction. Because these errors are not repeatable, they can never be eliminated. Empirically, however, it has been found that such random errors are frequently distributed according to a simple law. Therefore, it is possible to use statistical methods to deal with these random errors.

6.1.2 Types of Data

All data are not of the same type. When we use a scale of one to ten to rate the handling qualities of an aircraft, these data cannot be mathematically treated in the same way that we treat miss distance data on the bombing range. In fact there are four different types of data: nominal, ordinal, interval, and ratio data.

Nominal data are numerical in name only. If we refer to some standard aircraft configurations as configuration 1, 2, 3, or 4, we cannot treat these data with any of the normal arithmetic processes. For instance, we cannot say that 3 > l or that 3 – l = 2 or that 4 + 2 = 2. With nominal data, none of these arithmetic operations are applicable.

Ordinal data contains information about rank order only. If we rank order different aircraft by their maximum speed, then the resulting data can be compared to say that for example, 3 > 1 meaning aircraft three is faster than aircraft one. We cannot, however, say that 3 – 2 = 1, or that 4 + 2 = 2. Ordinal data can only be used to set up inequalities between the data.

Interval data contains both the rank order information of ordinal data, plus difference information. For example, temperature data has rank and difference information. If it is 30°F, 45°F, and 60°F at different times, the successive differences in temperature are the same, that is, 15°F. In both cases, the same amount of heat had to be added to raise the temperature by 15°F. We cannot say, however, that the end temperature of 60° is twice as hot as 30° even though 60° ÷ 30° = 2. The reason is that our zero point is arbitrary. Zero degrees Fahrenheit does not mean the absence of temperature. Thus, interval data has relative and difference information, but not ratio information.

Ratio data contains the information necessary to perform all the basic mathematical operations on the data. Most of our data falls into this category. Airspeed, fuel flow, range, etc., data all can be compared relatively, subtracted, and divided. We can legitimately say that a l000 NM range in one aircraft is four times as great as a 250 NM range in another.

This distinction between nominal, ordinal, interval, and ratio data is important. The type of data we have in a particular case may dictate the use of certain statistical techniques. But, before we can develop and use these statistical methods, we must first establish a common base of understanding of elementary probability.

6.1.3 Abbreviations and Symbols

The following unique symbols will be used in this text:

H_0	null hypothesis	z	standard normal deviate
H_1	alternate hypothesis	α	probability of type I error
n	number of samples	β	probability of type II error
$P(A)$	probability of event A	γ	efficiency of nonparametric test
s	sample standard deviation	δ	difference in means
U	rank sum statistic	μ	population mean
W	sign rank statistic	υ	degrees of freedom
$\bar{x}$	sample mean	σ	population standard deviation
$\tilde{x}$	sample median		
$\hat{x}$	sample mode		

6.2 Elementary Probability

A quantitative analysis of the random errors of measurement in flight testing (or any other experiment) must rely on probability theory. Probability theory is a mathematical structure which has evolved for the purpose of providing a model for chance happenings. The probability of an event is taken to mean the likelihood of that event happening. Mathematically, the probability of event A occurring is the fraction of the total times that we expect *A* to occur, or

$$P(A) = \frac{n_A}{N} \tag{6-1}$$

Where: $P(A)$ is the probability of *A* occurring.
n_A is the number of times we expect *A* to occur.
N is the total number of attempts or trials.

From this definition, it can be seen that $P(A)$ will lie between zero and one since the least that n_A can be is zero (*A* never happens), and the most it can be is N (*A* always happens).

In order to determine this fraction, n_A/N, we can approach the problem in two distinctly different ways. We can use our foreknowledge and make assumptions to predict the probability (classical or “a priori” probability) or we can conduct experiments to determine the probability (experimental or “a posteriori” probability).

6.2.1 Classical Probability

The study of classical probability began hundreds of years ago when games of chance became fashionable. There was much interest in questions about how frequently a certain type of card would be drawn or that a die would fall in a certain way. For example, it is almost obvious that if an ideal die (six sided) is honestly cast, there are six possible outcomes and the chance of getting a particular face number is one out of six; i.e., the probability is 0.16667.

The underlying conditions for simple evaluations such as this one are that:

1. every single trial must lead to one of a finite number of known possible outcomes, and
2. each possible outcome must be equally likely.

If we satisfy these two conditions, then the probability of event A is just

$$P(A) = \frac{n_A}{N}$$

Where now: n_A is the number of ways A can happen.

N is the total number of possible outcomes.

For example, what is the probability of getting no heads when we toss two fair coins? The possible outcomes are:

$$(H,H)\ (H,T)\ (T,H)\ (T,T)$$

Thus, $N = 4$ (that is four distinct, equally likely results) and $n_A = 1$ (only the result T,T has no heads). Therefore,

$$P(\text{no heads}) = 1/4 = 0.25$$

This approach to determining probabilities is instructive, but, in general, it is not applicable to experimental situations where the number of possible events is usually infinite and each possible outcome is not equally likely. Thus, we turn to experimental ('a posteriori') probability.

6.2.2 Experimental Probability

By definition, experimental probability is:

$$P(A) = \lim_{N \to \infty} \frac{n_{A_{obs}}}{N_{obs}} \tag{6.1}$$

where now: $n_{A_{obs}}$ is the number of times we observe A.

N_{obs} is the number of trials

For example, suppose we wish to check the classical result that the probability of getting a head when tossing a coin is 1/2. We toss the coin a large number of times and keep a record of the results. A typical graph of the results of such an experiment is shown in Figure 6.1. We will never, of course, reach an infinite number of trials, but our confidence in the probability of getting a heads will increase as the number of trials increases. As can be seen in Figure 6.1, the fraction of observed heads fluctuates dramatically when *N* is small, but as *N* increases, the probability steadies down to an apparently equilibrium value.

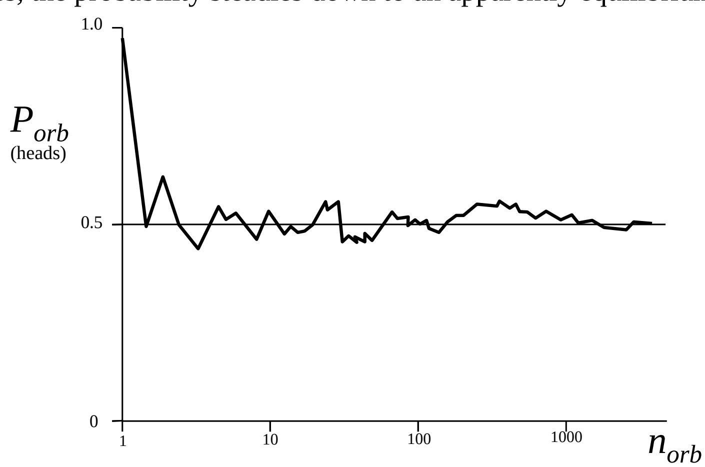


Figure 6.1 Experimental Probability

6.2.3 Probability Axioms

Probability theory can be used to describe the relationships between multiple events. Several axioms are presented. First, if the probability of *A* occurring is $P(A)$, then the probability of *A* not occurring, $P(\overline{A})$, is just: $P(\overline{A}) = 1 - P(A)$

This is easy to accept without a rigorous proof since the probability of something occurring has to be one. The remaining axioms presented below for multiple outcomes assume that each outcome is independent (*A* occurring does not subsequently affect the probability of *A* or *B* occurring) and mutually exclusive (only one can occur in a single trial). The two remaining axioms are:

$$P(A \text{ or } B) = P(A) + P(B)$$

$$P(A \text{ and } B) = P(A) \times P(B)$$

These axioms are also easily justified (as opposed to proven) by looking at classical probability. If we take the example of tossing a coin, then

$$P(H) = 0.5 \qquad P(T) = 0.5$$

and $P(H \text{ or } T) = 0.5 + 0.5$

which makes sense, because the probability of the coin coming up either heads or tails has to be one (excluding the chance of landing on edge).

Also, from the example of getting two tails in section 2.1,

$$P(T \text{ and } T) = P(T) \times P(T) = 0.5 \times 0.5 = 0.25$$

which is the same answer we got by examining all of the possible outcomes.

6.2.4 Probability Examples

Problem: Based on historical data, suppose we determine that 95% of the time an F-4 will make a successful approach end barrier engagement. If we have a flight of four that must use the barrier due to icy runway conditions, what is the probability that at least one aircraft will miss the barrier?

Solution: The probability that at least one will miss is the complement of the probability that all will successfully engage. That is

$$P(\text{1 or more miss}) = 1 - P(\text{all engage})$$

The probability that all four engage is

$$P(\text{all engage}) = P(\text{lst engages}) \times P(\text{2nd engages}) \times P(..... \qquad = P(\text{single engagement})^4$$

Finally, since $P(\text{single engage}) = 0.95$,

$$P(\text{1 or more miss}) = 1 - (0.95)^4 = 1 - 0.81 = 0.19$$

Or roughly speaking, about one out of five times, a flight of four F-4s would have at least one barrier miss.

Problem: What is the probability of getting craps (total of 2, 3, or 12) on a single roll of a pair of dice?

Solution: Since getting 2, 3, or 12 are independent, mutually exclusive events, we can use the following:

$$P(2, 3, \text{ or } 12) = P(2) + P(3) + P(12)$$

To get individual probabilities, first note that there are 36 possible outcomes (6^2) with two dice, each having six sides. In order to get a total of 2 or 12, there is only one way the dice can come up: 1 and 1, or 6 and 6, respectively. For a total of 3, the dice can come up two ways: 1 and 2, or 2 and 1. Therefore, since $P(A) = \frac{n_A}{N}$, we have:

$$P(2) = \frac{1}{36}$$

$$P(3) = \frac{2}{36}$$

$$P(12) = \frac{1}{36}$$

and finally, $P(2, 3 \text{or } 12) = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{9}$

Thus, about 11% of the time that you roll the dice, you will "crap" out.

6.3 Populations and Samples

6.3.1 Definition

Thus far in our discussion, we have made no distinction between populations and samples. The difference is an important one in the study of statistics. The definitions follow.

A population is all conceivable possible observations of a certain phenomena. Thus, many populations are infinite. For example, the population of the totals of two dice are all possible outcomes of rolling two dice, an infinite population. Another example, the population of weapon deliveries from an aircraft is all the possible drops it could make in its lifetime. A more limited population would be the scores of your class on the final exam. This population would have a limited number of observations, not infinity.

A sample is any subset of a given population. Thus the results of 1000 rolls of two dice constitute a sample of all possible results. The bomb scores from 100 weapon delivery sorties could be another example.

6.3.2 Assumptions

Constructing a population (what should be included as possibilities, what should be excluded?) or selecting a sample from a population must be done with care if we are later to apply statistical analysis techniques. The assumptions we normally impose on samples are that the data be homogeneous, independent, and random.

A homogeneous sample has data from one population only. If, for example, we allow bomb scores from an F-4C (iron sight) and an F-16C (predictive heads-up display) to be included in a single sample, the results would not be very meaningful.

An independent sample is one where the selection of one data point does not affect the likelihood of subsequent data points. For example, after dropping a bomb thirty feet long on the first pass, the probability that the next drop will miss by the same distance (or any other distance) is unaffected (independent). An example where the subsequent probabilities are affected is sampling from a finite population without replacement. For example, the probability of drawing a heart from a deck of cards changes if you sample and discard. The sample would remain independent if you replaced the card after each draw.

A random sample is one where there is an equal probability of selecting any member of the population. An example of a non-random sample would be using a single F-16 with a boresight error causing a bias in downrange miss distance to produce samples intended to be representative of all F-16 weapon deliveries.

6.3.3 Measures of Central Tendency

Given a homogeneous, independent, random sample, we now turn to methods to describe the contents of that sample. Suppose, for instance, we wish to be very accurate in measuring a hard steel rod with a micrometer. The population of measurements is all of the possible measurements that could be made with the micrometer. If we take a sample of ten measurements, we will probably get several different answers. The unpredictable variations could come from any of several different sources: we may tighten the micrometer more sometimes than others, there may be small dust particles sometimes, we may make small errors in estimating tenths of the smallest scale division, and so forth. Even so, we would expect to get a better answer by measuring many times rather than just once.

But what should we do with the multiple measurements, some of which are different? The most obvious procedure would be to average them. When we average the contents of a sample, we call the result the arithmetic mean, usually denoted by $\bar{x}$:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

The mean is the most common measure of central tendency, but not the only one. If we had taken 10 measurements and 8 of them were the same, we might feel justified in stating that this most common answer is the correct one and that the other 2 different answers were due to some unseen error. Using the most common sample is called taking the mode. The mode (usually denoted $\hat{x}$) is the most frequent sample value. In some samples, there may be more than one mode.

A third measure of central tendency is the median. The median (usually denoted $\tilde{x}$) value is the middle value. If we rank order the sample elements, then for an odd number of elements the median is just the middle value. For an even number of elements, we define the median as the arithmetic average of the two middle values.

Of the three different measures of central tendency (mean, mode, and median), the mean or arithmetic average is most commonly used.

6.3.4 Dispersion

Given that we typically use the mean as the single measure of central tendency of a sample, is that enough to adequately characterize the contents of a given sample? The answer is no. Using the mean by itself can be very misleading. For instance, consider the following two samples:

Sample l: 99.9, 100, 100.l

Sample 2: 0.l, 100, 199.1

As can be seen, the mean (and median in this case) is the same for both samples yet there is a significant difference between these two samples. The difference is in the variation of sample elements from the mean, or the dispersion. Thus, we now need some measure of the dispersion within a sample.

To obtain a measure of dispersion, first define the deviation, d_i, as the difference between the i^{th} element of the sample and the sample mean:

$$d_i \equiv x_i - \bar{x}$$

The first inclination may be to average these deviations, but the result is not illuminating since:

$$\bar{d}_i = \frac{1}{N}\sum_{i=1}^{N} d_i = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})$$

$$= \frac{1}{N}\sum_{i=1}^{N} x_i - \bar{x} = \bar{x} - \bar{x} = 0$$

Because of the definition of the mean, the deviations above the mean will always exactly cancel the deviations below the mean. This result may lead you to conclude that we should average the absolute values of the individual deviations. Doing so produces what is referred to as the mean deviation:

$$\text{mean deviation} \equiv \frac{1}{N}\sum_{i=1}^{N}\left|x_i - \bar{x}\right|$$

This quantity is sometimes used as a measure of dispersion, but for reasons that will become apparent later, a more common measure of dispersion is the standard deviation, which is defined next.

In defining the standard deviation, we eliminate the negative individual deviations by squaring each term, rather than by taking the absolute values. We then average the squares and finally take the positive square root of the results. Thus, the standard deviation (denoted by σ) is the root-mean-square deviation:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N} d_i^2}$$

The square of the standard deviation, σ^2, is called the variance.

6.3.5 Notation

Normally, we use Greek letters to denote statistics (such as mean and variance) for populations and we use Roman letters for statistics of samples. Therefore, we will use:

μ and σ^2 for population mean and variance

$\bar{x}$ and s^2 for sample mean and variance

There is one other difference between population and sample statistics. The sample standard deviation is defined slightly differently than the population standard deviation:

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2}$$

The difference is that the sum of the squares is divided by $N - 1$ for the sample rather than by just N as for the population standard deviation. This is explained by mathematicians as being due to losing a degree of freedom. The effect is to make the sample standard deviation slightly larger than it would have been and the difference decreases as the sample gets larger.

6.3.6 Example

Problem: Given the following 10 observations find the sample mean, median, mode, and standard deviation: (3, 4, 6, 6, 6, 8, 9, 10, 12, 15)

Solution:

$$\bar{x} = \tfrac{1}{10}(3 + 4 + 6 + 6 + 6 + 8 + 9 + 10 + 12 + 15) = 7.9$$

$$\hat{x} = 6 \text{ (Most Frequent)}$$

$$\tilde{x} = \tfrac{1}{2}(6 + 8) = 7 \text{ (average of two middle values)}$$

$$s = \sqrt{\tfrac{1}{9}\left(4.9^2 + 3.9^2 + 1.9^2 + 1.9^2 + 1.9^2 + 0.1^2 + 1.1^2 + 2.1^2 + 4.1^2 + 7.1^2\right)}$$

$$s = 3.695$$

6.4 Probability Distributions

Now that we have covered elementary probability concepts and introduced the idea of population and samples, we turn to probability distributions. Application of statistical methods requires an understanding of the characteristics of the data obtained. Probability distributions, either empirical or theoretical, can give us these required characteristics. Most statistical methods are based on theoretical distributions which approximate the actual distributions.

6.4.1 Discrete Probability Distributions

To introduce the idea of a probability distribution, let's go back to the example of tossing two coins introduced earlier. We can calculate from classical probability the probability of getting 0, 1, or 2 heads. Tabulating this as $f(n)$, where n is the number of heads obtained on a single toss of two coins:

n	*f*(n)
0	0.25
1	0.5
2	0.25

Table 6.1 Probability of Getting *n* Heads in Two Tosses of a Fair Coin

Another method of presenting this data would be graphically by means of a bar graph, as shown in Figure 6.1.

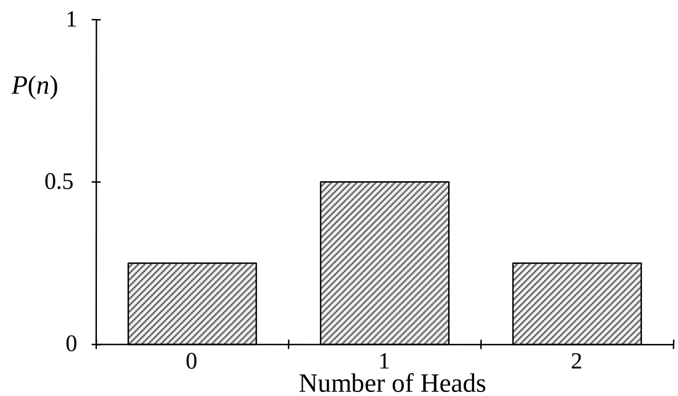


Figure 6.1 Probability of Getting *n* Heads in Two Tosses of a Fair Coin

Thus, $f(n)$ is called the *probability distribution* of n. The above example is a theoretical calculation. More frequently, we are concerned with empirical distributions. For example, suppose we collect a sample of data on an aircraft's landings as shown in Table 6.2.

Touchdown Distance	Frequency in	Relative Frequency

from Aim Point	Distance Interval	
0 - 100 ft	2	0.05
101 - 200 ft	10	0.25
201 - 300 ft	18	0.43
301 - 400 ft	8	0.20
401 - 500 ft	3	0.07

Table 6.2 Touchdown Data

Plotting the data in a histogram as in Figure 6.2 will give us a graphical representation of this empirical probability distribution.

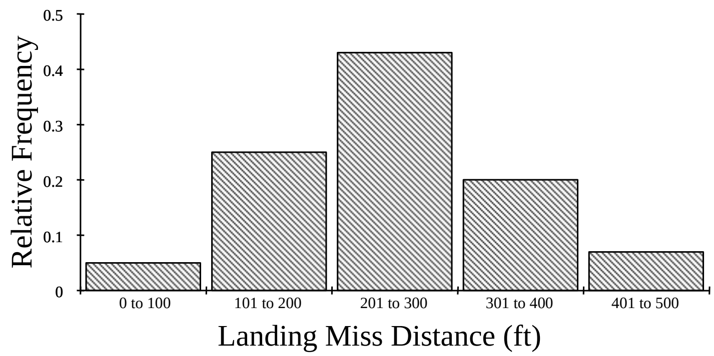


Figure 6.2 Probability Distribution of Touchdown Miss Distance

6.4.2 Continuous Probability Distribution

If we acquire more landing data and reduce the size of the intervals, we could draw a new histogram. In the limit as we acquire more and more data, and reduce the interval size to smaller and smaller values, the histogram approaches a smooth curve, as shown in Figure 6.3.

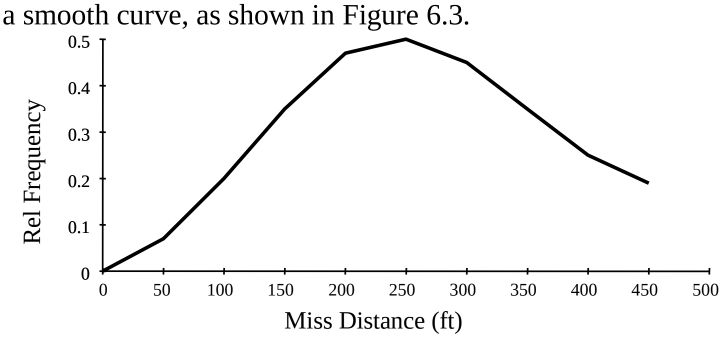


Figure 6.3 Continuous Probability Distribution of Touchdown Miss Distance

This smooth, continuous probability distribution cannot be interpreted in the same way as the discrete distribution. In Figure 6.2, the height of the bar above the interval is the probability that *x* will have a value within that interval. In Figure 6.3, the height of the curve above a point is not the probability of x having that point value. Since there are an infinite number of points, (i.e., a continuous curve) the probability of *x* having any single specific value is zero. We can, however, talk about the probability of *x* being between two points, *a* and *b*. Then, the interpretation of the continuous probability distribution is as follows:

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

That is, the probability that *x* falls between *a* and *b* is the area under the probability distribution curve between *x* = *a* and *x* = *b*, as shown in Figure 6.4.

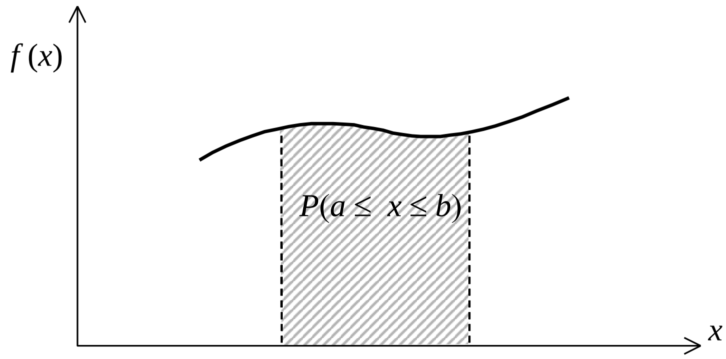


Figure 6.4 Probability as the Area Under a Continuous Probability Distribution

From this, we can see that $f(x)$ must always be greater than or equal to zero. Negative areas would be meaningless. Also, since the maximum probability is one, we have:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

6.4.3 Cumulative Probability Distribution

For some applications, displaying the probability distribution as a cumulative function is the most useful method. A cumulative probability distribution gives the probability that a random variable x is equal to or less than a given value, a. In mathematical terms:

$$F(x) = P(x \le a) = \int_{-\infty}^{a} f(x)dx$$

For example, the relative probability of aircraft landing miss distances from Figure 6.3 could be displayed in a cumulative distribution as in Figure 6.5.

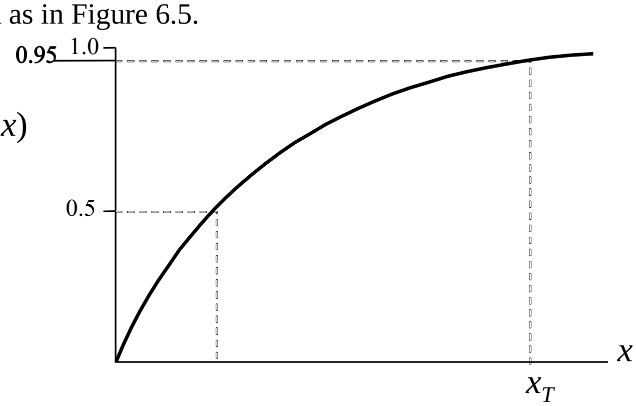


Figure 6.5 Cumulative Probability Distribution

From this type of display, the median, x, can be directly read. Also, we can see that 95% of the time we expect the miss distance to be below some value, x_T.

6.4.4 Special Probability Distributions

There are numerous theoretically derived probability distributions used in analyzing data. In this course, we will limit our scope to only four distributions: binomial, normal, student's t, and χ^2. Each is briefly introduced below.

6.4.5 The Binomial Distribution

The first special probability distribution that we will examine is a discrete probability distribution, the binomial distribution. The binomial distribution is a theoretically derived distribution of probabilities for trials in which there are two possible results, usually called success and failure. This can be applied to a large number of problems if success and failure are defined beforehand, for example:

1. Toss of a coin - heads (success) or tails (failure).
2. Roll of two dice - total of 7 (success) or other than 7 (failure).
3. Qualitative evaluation of a flight control modification - better (success) or worse (failure).

Determining the probability of getting exactly n successes in N trials given the probability of a single success is our objective. Let p represent the probability of a single success. First, the limiting cases are very simple. If $n = N$, then the probability is just p^n. If $n = 0$ (all failures), then the probability is simply $(l - p)^N$, or, if we let $l - p = q$, then q^N.

The in between probabilities are not as simple. If we have n successes and $N - n$ failures, we might be tempted to say the $p^n q^{N-n}$ is the probability, but there are multiple combinations of n objects possible in N events. Luckily, mathematicians have quantified how many combinations are possible and the probability of exactly n successes in N trials is:

$$f(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}$$

where $x! \equiv x(x-1)(x-2)\ldots(3)(2)(1)$

An example may help illustrate. If two different flight control systems are really equally desirable, then the probability of 6 out of 8 pilots preferring system A over system B can be found using the binomial distribution. If A and B are truly equally good, the probability of a pilot picking A over B is equal to 1/2 ($P = q = 0.5$). The probability of 6 out of 8 picking A is

$$f(6) = \frac{8!}{6!2!}(0.5)^6(0.5)^2 = 0.109$$

Thus, if you assumed that A and B were equally good, then there is only an 11% chance of getting the test results you observed, implying that your initial assumption may be in error. In a similar way, the probabilities for all possible results can be graphed as shown in Figure 6.6.

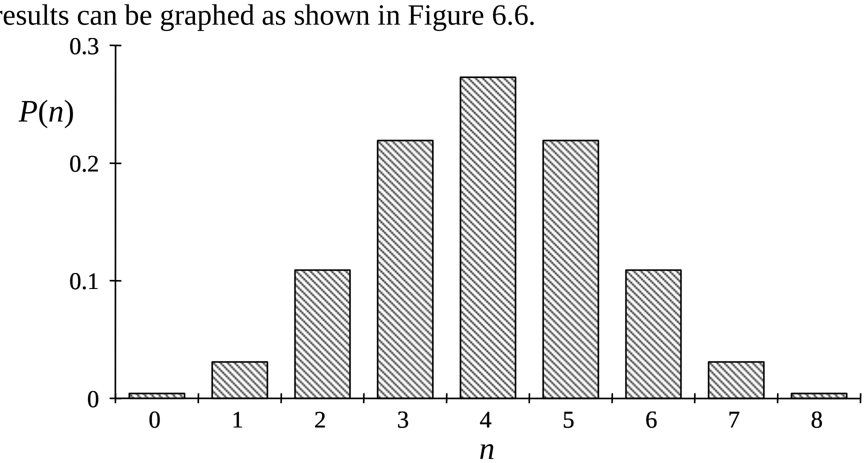


Figure 6.6 Probability that n of 8 Pilots Will Prefer System A if $p = q = 0.5$

6.4.6 The Normal Distribution

The normal distribution is the single most important distribution in data analysis. The theoretical basis for the normal distribution lies in the binomial distribution. If we consider any deviation from the mean as the result of a large number of elemental errors, all of equal magnitude and each equally likely to be positive or negative, we can derive the following:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Thus, the normal distribution is a continuous probability distribution, valid from $-\infty < y < \infty$. Its graphical representation is shown in Figure 6.7.

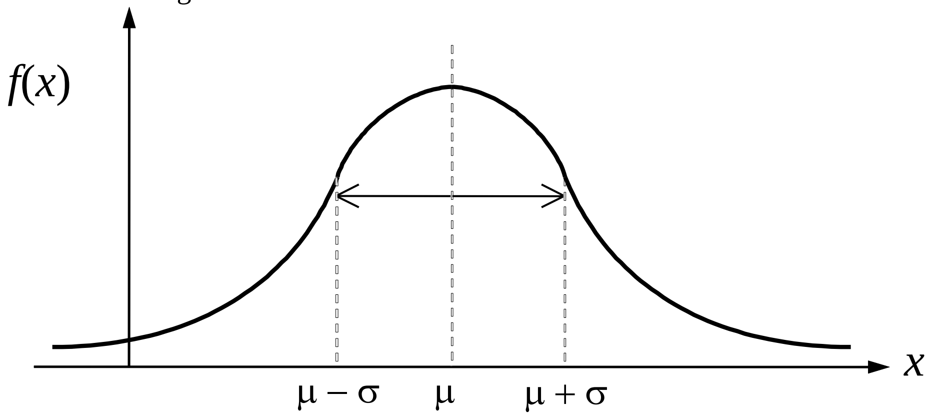


Figure 6.7 Normal Probability Distribution

From this figure, it can be seen that $f(x)$ is symmetric about $x = \mu$, that $x = \mu$ yields the maximum value of $f(x)$. Also, $x = \mu \pm \sigma$ are the two points of inflection on the curve of $f(x)$.

Notwithstanding the mathematical derivation of the normal distribution from a binomial distribution, the most compelling justification for its use and study is the fact that many sets of experimental observations have been shown to obey it. Accordingly, the distribution has been studied extensively.

Recalling that for a continuous probability distribution, the probability that x lies between a and b is defined by the integral of $f(x)$ between a and b, we come to a major drawback of the normal distribution. For example, what is the probability of getting $x < a$ if x is normally distributed? Just:

$$P(a < x < b) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

which cannot be solved in closed form. Numerical techniques are required. Tables could be used except different tables would be needed for each combination of μ and σ. The problem is overcome by making a substitution of variables in $f(x)$ by letting

$$z = \frac{x-\mu}{\sigma} \quad \text{and} \quad \frac{dz}{dx} = \frac{1}{\sigma}$$

so that now:

$$P(z < a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz$$

Tables are abundant for $f(z)$ which is, in effect, the normal distribution with a mean of zero and a standard deviation of one. To use these standardized normal tables, we must simply change our variable x to z as shown above. Values for $f(z)$ are tabulated in the appendix.

A graph of the standard normal distribution curve, with approximate percentages under the curve is given in Figure 6.8.

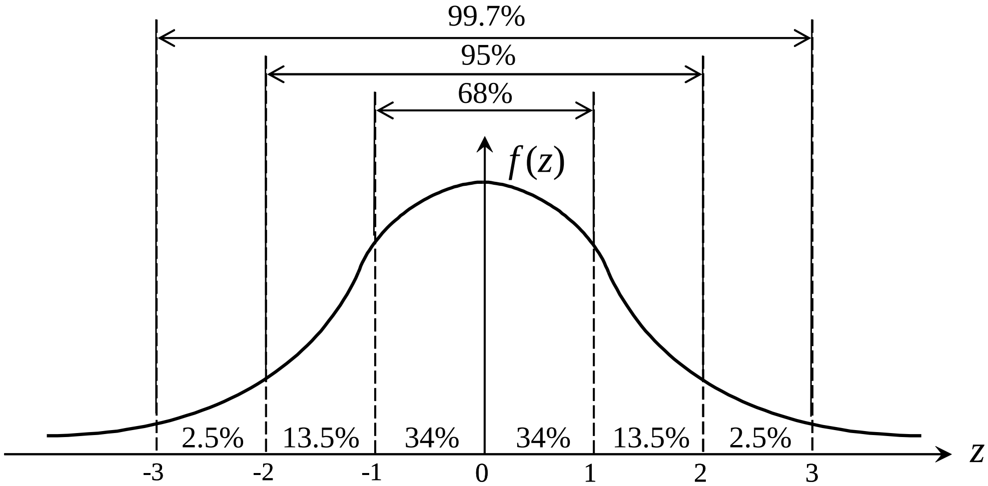


Figure 6.8 Standardized Normal Distribution

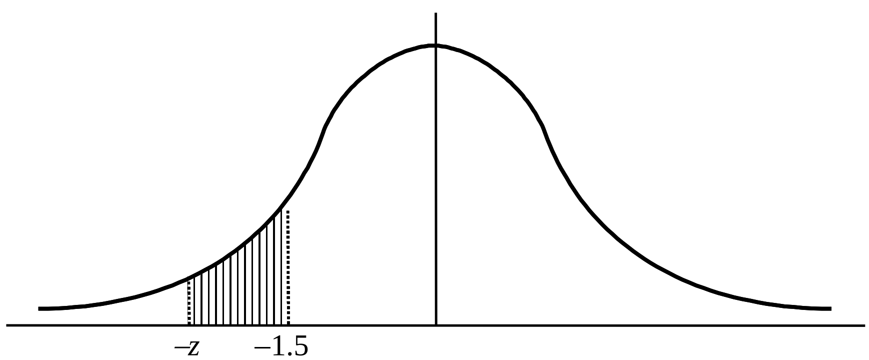
The following three examples may help illustrate the meaning of the normal distribution and the uses of the standardized tables:

Normal Distribution Examples

- Find the area between $z = 0.81$ and $z = 1.94$. Using the standard normal distribution table in Appendix A-1, proceed down the column marked z until entry 1.9 is reached, then right to the column marked 0.04. The result, 0.9738, is the area between $-\infty$ and 1.94. Similarly, 0.7910 is the area from $-\infty$ to 0.81. If we subtract these two values,

$$P\ (0.81 < z < 1.94) = 0.9738 - 0.7910 = 0.1828$$

- Find the value of z such that the area between –1.5 and z is 0.0214. (Assume z is negative but the left of -1.5.)



Area between $-z$ and 1.5 = (area between –1.5 and $-\infty$) - (area between $-z$ and $-\infty$)

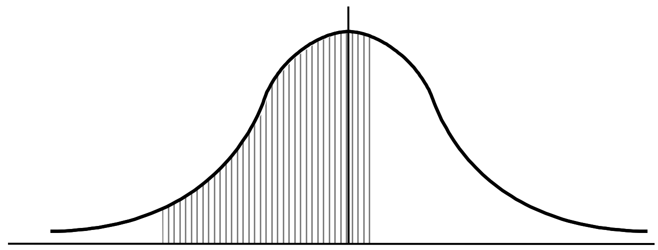
0.0214 = 0.0668 - (area between $-z$ and $-\infty$)

$\therefore \quad z = -1.69$

- The mean fuel used for a given profile flown 40 times was 8000 lbs, and the standard deviation was 500 lbs. Assuming the data is normally distributed, find the probability of the next sortie using between 7000 and 8200 pounds.

$$\text{7000 lbs in standard units } = \frac{x-\mu}{\sigma} = \frac{7000-8000}{500} = -2$$

$$\text{8200 lbs in standard units } = \frac{8200-8000}{500} = 0.4$$



$P(-2 \le z \le 0.4)$ = (area between $z = -\infty$ and $z = 0.4$) - (area between $z = -\infty$ and $z = -2$)
= 0.6554 – 0.0228 = 0.6326

6.4.7 The Student's *t* Distribution

In order to use the standard normal distribution, we must know the population mean and standard deviation. In practical applications, we frequently do not know these values and instead must use the sample mean and standard deviation. The difference between the sample mean and true mean of a population was investigated first by W. S. Gossett. He developed a theoretical distribution for the statistic

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

where *t* is used as a measure of the difference between the sample mean and the true mean. As can be seen, the value of *t* is also influenced by how much dispersion we have in our sample and by the size of that sample.

For each possible value of *n*, we can plot a probability distribution of *t*. The distribution looks very similar to the standard normal distribution, especially for large values of *n*. In fact, it can be shown mathematically that as *n* approaches ∞, the *t* distribution approaches the normal distribution. Figure 6.9 compares *t* distributions for different values of *n*.

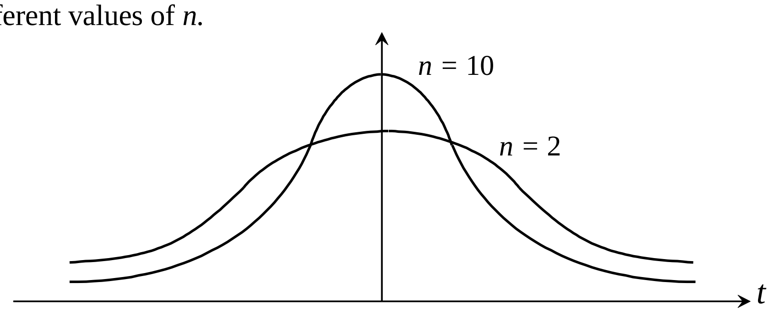


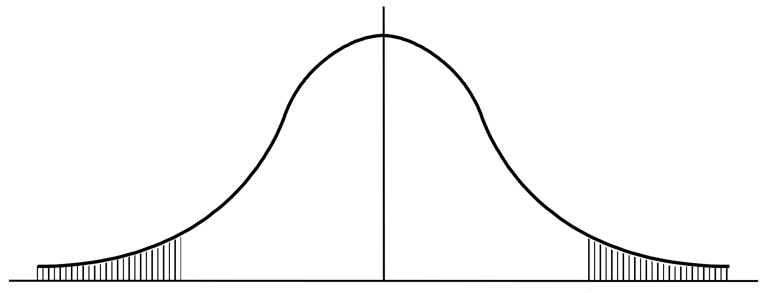
Figure 6.9 Change in *t*-Distribution with Sample Size

Because of this change in *t* with sample size, different t distributions must be tabulated for each value of *n*. Typically, as in the appendix, different critical values of $f(t)$ are tabulated for different values of *n* up to about $n = 30$, beyond which one could use the standard normal distribution with $\bar{x} = \mu$ and $s = \sigma$ with very little error. It should be noted that most tables use degrees of freedom, *v*, instead of *n*, where

$$v = n - 1$$

The theoretical reasons for this change are of little consequence to us here.

t - Distribution Examples



- Find the t_1 for which the total shaded area on the right = 0.05, if we assume 9 degrees of freedom. If the area on the right of t_1 = 0.05, then the area to the left is (l – 0.05) = 0.95 and t represents the 95th percentile, $t_{0.95}$. Referring to the student's t - distribution table in Appendix A-2 proceed down the column headed ν until reaching 9. Then proceed right to column headed $t_{0.95}$. The result 1.83 is the required value of t.
- Find the t_1, for which the total shaded area = 0.05, assuming 9 degrees of freedom. If the total shaded area is 0.05, then the shaded area on the right is 0.025 by symmetry. Thus, the area to the left of is (l – 0.025) = 0.975 and t_1 is $t_{0.975}$. From the appendix, we find 2.26 as the required value of t.

6.4.8 The Chi-Squared Distribution

Just as the sample mean differs from the population mean, we expect the sample standard deviation to differ from the true population value. The difference is distributed according to the Chi-squared distribution of the statistic

$$\chi^2 = \frac{(n-1)\,s^2}{\sigma^2}$$

which is a measure of the dispersion of experimental s values around the population value, σ, caused by taking only limited sample sizes. A sketch of the Chi-square probability distribution is shown in Figure 6.10.

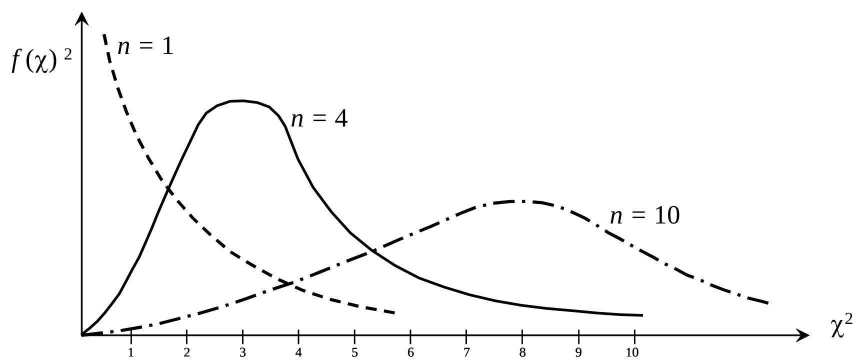
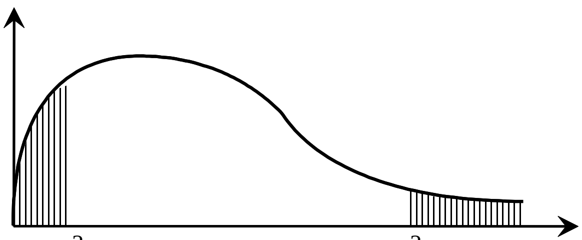


Figure 6.10 Change in χ^2 distribution with sample size

As with the t distribution, the χ^2 distribution changes with sample size and therefore critical values of χ^2 are normally tabulated (as in the appendix) for various degrees of freedom (n –1).

Chi-Square Distribution Examples



- Find the value of χ_2^2 for which the shaded area on the right = 0.05 assuming 5 degrees of freedom. If the shaded area on the right is 0.05, then the total area to the left of χ_2^2 is (1 – 0.05) = 0.95 and χ_2^2 represents the 95th percentile, $\chi_{0.95}^2$. Referring to the χ^2 distribution table in Appendix A-3, proceed down the *v* column until entry 5 is reached. Then proceed right to the column headed $\chi_{0.95}^2$. The result, 11.1 is the required value of χ_2^2.
- Find χ_1^2 and χ_2^2 for which the total shaded area = 0.05, assuming 5 degrees of freedom. Since the distribution is not symmetric, there are many values for which the total shaded area = 0.05. It is customary, unless otherwise specified, to choose the two areas equal. In this example, then, each area = 0.025. If the shaded area on the right is 0.025, the area to the left of χ_2^2 is $(1-0.025)=0.975$ and χ^2 is the 97.5th percentile, $\chi_{0.975}^2$ which from the appendix is 12.8. Similarly, if the shaded area on the left is 0.025, the area to the left of χ_1^2 is 0.025, and χ_1^2 represents the 2.5th percentile, $\chi_{0.025}^2$ which equals 0.831.
- Find the median value of χ^2 corresponding to 28 degrees of freedom. Using the table in the appendix, we find in the column headed $\chi_{0.5}^2$ (since the median is the 50th percentile), the value is 27.3 corresponding to $v = 28$.

6.5 Confidence Limits

In practical situations, we normally take a sample of a large population such as takeoff distance or bomb miss distance, and we use the mean of our multiple observations as a point estimate of the true population mean. We often report this sample mean as though it were the true answer. We must realize, however, that any subsequent single observation can be expected to differ from our sample mean and that the true population mean may differ from our sample mean. If we design the test correctly (standardize the method and conditions) and take sufficient samples (to be discussed in a later section), we will have confidence that our answer is sufficiently accurate. There exist quantitative methods for determining how far away our answer is likely to be from the true answer (a confidence interval). These methods are the subject of this section.

6.5.1 Central Limit Theorem

The central limit theorem is required to establish confidence limits on both the population mean and standard deviation. The central limit theorem can be stated as follows:

Given a population with mean, μ, and variance, σ^2, then the distribution of successive sample means, from samples of *n* observations, approaches a normal distribution with mean, μ, and variance σ^2/n.

In simpler terms, if we start with a general population *A*, where the mean is μ_A and the variance is σ_A^2, and take multiple samples each of size *n*, then the resulting sample means will also have some distribution with a mean and variance $\left(\mu_{\bar{x}}, \sigma_{\bar{x}}^2\right)$. Regardless of the original distribution of *A*, the distribution of the means will be approximately normal (it gets better as *n* is increased). Also, the mean of the means will be the same as the mean of *A*. And, finally, the variance of the means is the variance of *A* divided by *n*. This is depicted in Figure 6.1.

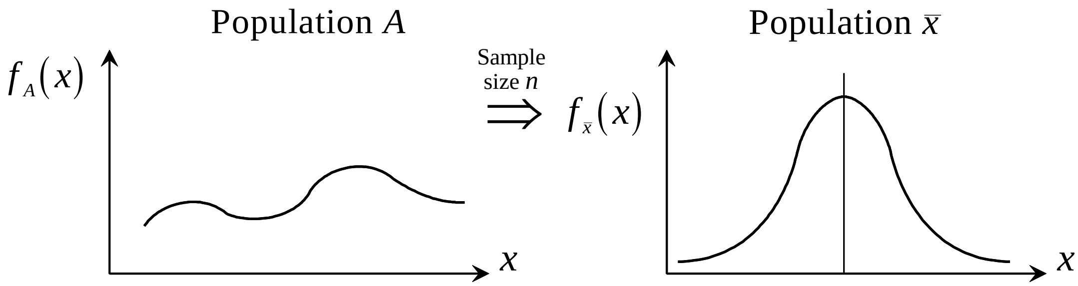


Figure 6.1 Central Limit Theorem

Although proof of the central limit theorem is beyond our scope here, a cursory inspection shows that it passes the common sense test. If our sample size is very small (say l), then for many samples, the distribution of our means is identical to the original and $\mu_{\bar{x}} = \mu_A$ and . $\sigma_{\bar{x}} = \sigma_A$ At the other extreme, if n is infinite (exhaustive) then we always get the true population mean and variance. Accordingly, $\mu_{\bar{x}} = \mu_A$ and $\sigma_{\bar{x}} = 0$. We now turn to using the central limit theorem to establish confidence intervals.

6.5.2 Confidence Interval for the Mean

If we take a sample of size n, we now know that the distribution of the means of multiple samples would be approximately normally distributed, as shown in Figure 6.2.

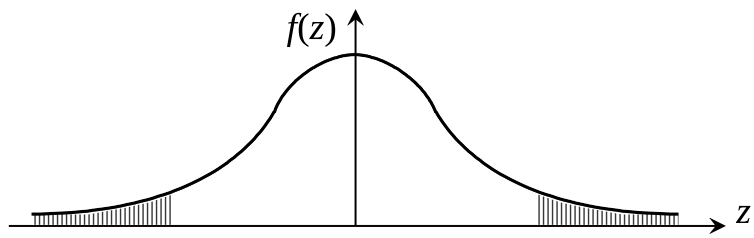


Figure 6.2 Establishing Confidence Limits on the Mean

From the definition of a normal probability distribution, we can say that a sample *z* will be between $-z_{1-\alpha/2}$ and with probability $1 - \alpha$, or

$$p\left(-z_{1-\alpha/2} < z < z_{1-\alpha/2}\right) = 1 - \alpha$$

If our *z* comes from one of the sample means, $z = \dfrac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}}$

or, using the central limit theorem $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

thus
$$p\left(-z_{1-\alpha/2} < \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} < +z_{1-\alpha/2}\right) = 1 - \alpha$$

or
$$p\left(\bar{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

That is, $(1 - \alpha)$ 100% of the time, the true population mean, μ, will be within $\pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$ of the sample mean. The range of values is the interval and $(1 - \alpha)$ is the confidence level.

As an example, suppose we wanted to know the 95% and 99% confidence intervals for the maximum thrust of new F-100 engines given that a sample of 50 engines produced a mean max thrust of 22,700 lbs with a sample standard deviation s = 500 lbs.

l. At 95%, $\alpha = 0.05$ and $z_{1-\alpha/2} = 1.96$.

$$\mu = \bar{x} \pm z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Therefore $$\mu = 22{,}700 \pm 1.96 \frac{500}{\sqrt{50}}$$

or $22{,}561 < \mu < 22{,}839$

2. At 99%, $\alpha = 0.01$ and $z_{1-\alpha/2} = 2.58$.

Thus $$\mu = 22{,}700 \pm 2.58 \frac{500}{\sqrt{50}}$$

or $22{,}518 < \mu < 22{,}882$

The above examples point out two important considerations. As you might have anticipated, as the requirement for certainty increases (95 to 99%), the interval widens. Given that the normal probability is continuous from $-\infty$ to $+\infty$, if we require that we be 100% certain that the true m falls within our interval, the confidence interval becomes meaningless: $-\infty < \mu < +\infty$, or put another way, if you want to be absolutely certain you're right, you can't say you know anything.

The second important point is that to construct the interval we had to use *s* as an estimate of σ. This is, in fact, a legitimate estimate if $n > 30$. For smaller sample sizes, we cannot make this assumption and must resort to the method described in the next section.

6.5.3 Confidence Interval for Mean for Small Samples

When the sample size is less than 30 and the population variance is unknown (the typical case in flight testing), we must substitute *t* (defined earlier) for *z*:

$$\left(\bar{x} - t_{\nu,1-\alpha/2} \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\nu,1-\alpha/2} \frac{s}{\sqrt{n}} \right)$$

As an example, suppose our earlier data on F-100 engines was based on a sample of only 5 engines. Then at the 95% confidence level:

$$\frac{\alpha}{2} = 0.025 \text{ and } \nu = 4, \text{ thus } t_{4,\,0.975} = 2.78$$

and $$\mu = \bar{x} - t_{\nu,1-\alpha/2} \frac{s}{\sqrt{n}} = 22{,}700 \pm 2.78 \frac{500}{\sqrt{5}}$$

or $22{,}078 < \mu < 23{,}321$

And as you should have expected the interval at the same confidence level had to increase to accommodate the smaller sample size.

6.5.4 Confidence Interval for Variance

In a manner similar to that of confidence intervals for the mean, we can establish a confidence interval for the variance based on the previously defined statistic χ^2:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

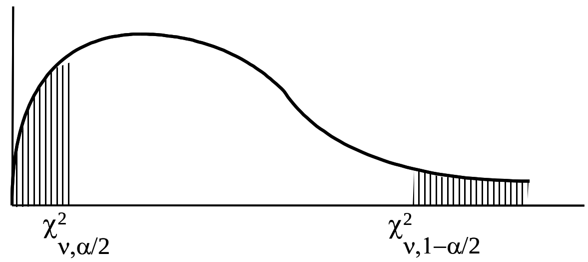


Figure 6.3 Establishing Confidence Limits on Variance

From Figure 6.3, we can see that the probability of our sample statistic χ^2 falling between $\chi^2_{\nu,\alpha/2}$ and $\chi^2_{\nu,1-\alpha/2}$ is just $1 - \alpha$.

$$P\left(\chi^2_{\nu,\frac{\alpha}{2}} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\nu,1-\frac{\alpha}{2}}\right) = 1-\alpha$$

thus, with $(1 - \alpha)$ 100% confidence,

$$\chi^2_{\nu,\frac{\alpha}{2}} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\nu,1-\frac{\alpha}{2}}$$

or

$$\frac{(n-1)s^2}{\chi^2_{\nu,1-\frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{\nu,\frac{\alpha}{2}}}$$

For example, if we take a sample of size 6 and find that the sample standard deviation is 2, we can specify with 95% probability between what limits the true population variance lies. In this case, we have:

$$\frac{\alpha}{2} = 0.025,\ 1-\frac{\alpha}{2} = 0.975,\ \nu = 5,\ s = 2$$

thus

$$\frac{(6-1)2^2}{\chi^2_{5,0.975}} < \sigma^2 < \frac{(6-1)2^2}{\chi^2_{5,0.025}}$$

where $\chi^2_{5,0.975} = 12.8$, $\chi^2_{5,0.025} = 0.831$

thus

$$\frac{(5)4}{12.8} < \sigma^2 < \frac{(5)4}{0.831}$$

or

$$1.56 < \sigma^2 < 24.1$$

The large band is due to the small sample size. If the sample variance were the same for a larger sample (say $n = 18$), then the confidence interval would be smaller, for instance

$$\frac{(17)4}{30.2} < \sigma^2 < \frac{(17)4}{7.56}$$

Or

$$2.25 < \sigma^2 < 8.99$$

6.6 Hypothesis Testing

Closely tied to the idea of confidence intervals is perhaps the most important part of statistical analysis: hypothesis testing. A statistical hypothesis is a statement, which may or may not be true, concerning one or more populations. Instead of using our sample data to make a point or interval estimate of some population parameter, we first hypothesize that a population parameter is such and such, and then use sample data to determine the reasonableness of our hypothesis. The truth or falsity of a statistical hypothesis is never known with absolute certainty unless we examine the entire population. This is certainly the case in nearly all flight tests. A simple example may illustrate the concept.

Suppose we assume (hypothesize) that a given coin is fair, that is, the probability of heads is 0.5. To determine if our assumption is correct we toss the coin 100 times. If the results are 48 heads, we may conclude that it is reasonable to say the coin is fair. If, on the other hand, we get only 35 heads, it may be more reasonable to conclude that the coin is not fair. The subject of this section is how to draw the line in cases like this.

6.6.1 Null and Alternate Hypotheses

It should be emphasized at the outset that the acceptance of a statistical hypothesis is a result of insufficient evidence to reject it and does not necessarily mean that it is true. Because of this fact, we must be careful in setting up our hypothesis since, in the absence of data, we will be forced to accept our original hypothesis. Usually, we select this hypothesis with the sole objective of rejecting (or nullifying) it. Hence, it is called the null hypothesis, denoted H_0. The null hypothesis is usually formulated so that in the case of insufficient data, we return to the status quo or safe conclusion. Examples of null hypotheses are:

1. The defendant is innocent (not a statistical hypothesis, but a good illustration).
2. The lock-on range of a new RADAR is no better than that of the present RADAR.
3. The MTBF of a new part is no better than that of the existing part.

Since we are attempting to negate our null hypothesis, we should have established an alternate hypothesis, denoted H_1 to reflect what we want to prove and let H_0 then be the negation of H_1.

Examples:

1. H_0: $\mu = 15$ H_1: $\mu \neq 15$
2. H_0: $p \geq 0.9$ H_1: $p < 0.9$
3. H_0: $\mu_1 = \mu_2$ H_1: $\mu_1 \neq \mu_2$

6.6.2 Types of Errors

Regardless of how carefully we set up a test, there is always the chance that we will come to the wrong conclusion. In our earlier example of tossing a coin assumed to be fair, the result of 35 heads out of 100 times could be simply due to chance variation of a fair coin (the probability of this occurring is small, 0.0026, but not zero). If we reject the null hypothesis when in fact it is true, this is called a Type I error, and the probability of doing so is denoted α, called the level of significance.

A different error results if we accept the null hypothesis when it is false. This is a Type II error, and its probability is denoted by β. For example, if in the coin experiment, we concluded it was a fair coin based on a result of 43 heads out of 100, the coin may really have a probability of heads of .4 and the 48 result was due to chance variation (in this case $\beta = 0.10$).

Generally, because of the fail-safe wording of the null hypothesis, we desire to have α, the probability of rejecting H_0 when it is true, very small, usually 0.05 (occasionally 0.01). The smaller α is, however, the larger β becomes. Generally, β is larger than α since this is a more acceptable error (a large β implies we stay with the status quo, H_0, more frequently than we should). The only way to reduce both α and β is to take more data. If we do exhaustive sampling, α and β go to zero.

6.6.3 One Tailed vs Two Tailed Tests

During some tests, we are interested in extreme values in either direction. Burn times on rocket motors might be an example. Too long or too short of a burn time may have dire consequences for system performance. For tests of this sort, we would form hypothesis of the form:

$$H_0: \mu = \mu_0 \quad \text{and} \quad H_1: \mu \neq \mu_0$$

In these cases, we should reject H_0 whenever our sample produced results that were either too high or too low. Thus, our level of significance, α, would be divided into two equal regions as shown in Figure 6.1(b). In most flight test examples, however, we are concerned with extremes in one direction only. For example, we hypothesize that the aircraft meets the contractual specification for takeoff distance. The only significant alternative hypothesis is that the actual takeoff distance is longer than the specification. For tests of this sort, we would form hypothesis of this form:

$$H_0: \mu \leq \mu_0 \quad \text{and} \quad H_1: \mu > \mu_0$$

or H_0: $\mu \geq \mu_0$ and H_1: $\mu < \mu_0$

In these cases, we would reject H_0 only when our sample produced results that were extreme in one direction. Thus, our level of significance, a, would be in one tail of the curve only as showing Figure 6.1(a).

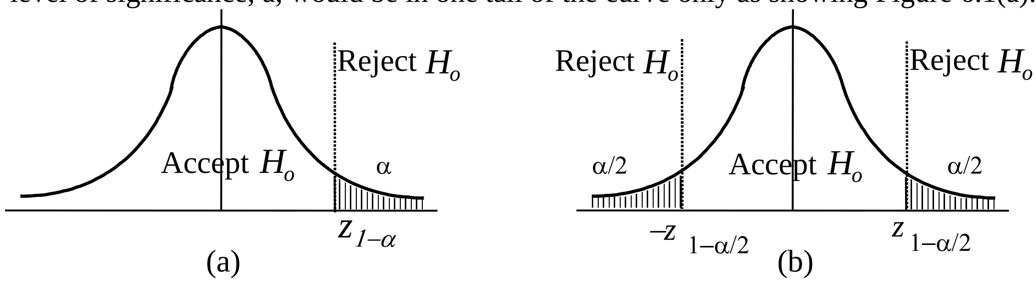


Figure 6.1 One-Tailed vs Two-Tailed Test

6.6.4 Tests on Means

The first step in hypothesis testing is to formulate the null and alternate hypothesis. Second, choose the level of significance (a) and define the areas of acceptance and rejection. Third, collect data and compare the results to what was expected. Fourth, accept or reject the null hypothesis. For tests on means, we will use the same statistic we used in constructing confidence intervals:

For $n > 30$ or σ known, use $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

For $n < 30$ and σ unknown, use $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$

The following two examples should illustrate the method:

Example 1: Two tailed test on mean, σ known. During early testing of the F-19 bombing system, it was determined that the cross range errors for 30° dive bomb passes were normally distributed with a mean error of 20 feet and a standard deviation of 3 feet. After a flight control modification to reduce adverse high AOA flying qualities, it was found that the mean cross range error for nine bomb runs was 22 feet. Has the mean changed at the 0.05 level of significance?

Step one: Form null and alternate hypothesis:

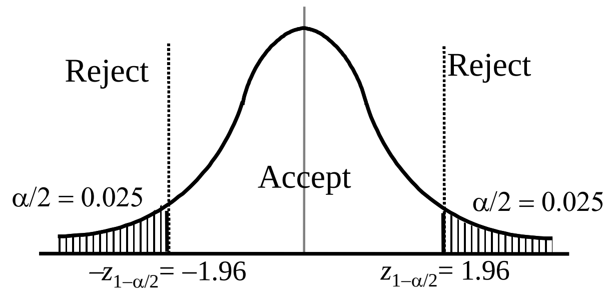
$$H_0: \mu = 20 \text{ (status quo)} \quad \text{and} \quad H_1: \mu \neq 20$$

Step two: $\alpha = 0.05$ (given) and this will be divided into two tails, high and low, since extreme values in either direction would indicate that μ has changed.

Step three: Since σ was not given, we will assume that σ has not changed significantly from the unmodified system. This is not an obvious truth, but its use here illustrates the criteria for using the *z* statistic. In any case, our data gives:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{22 - 20}{3/\sqrt{9}} = 2$$

Compare this to the areas of rejection/acceptance below



Step four: Because $z > z_{1-\alpha/2}$ (2 > 1.96) we must reject the null hypothesis and conclude that (with 95% confidence) the mean cross range bombing error has changed due to the flight control modification.

Example 2: One tailed test on mean, small sample, σ unknown. Suppose we fly nine sea level to 20,000 ft PA check climbs to verify a contract specification which states that the fuel used in this climb shall not be greater than 1,500 pounds. We find that our sample of nine climbs used an average of 1,600 pounds with a sample standard deviation of 200 pounds.

Step one: Form null and alternate hypothesis:

H_0: $\mu \le 1500$ (innocent until proven guilty)

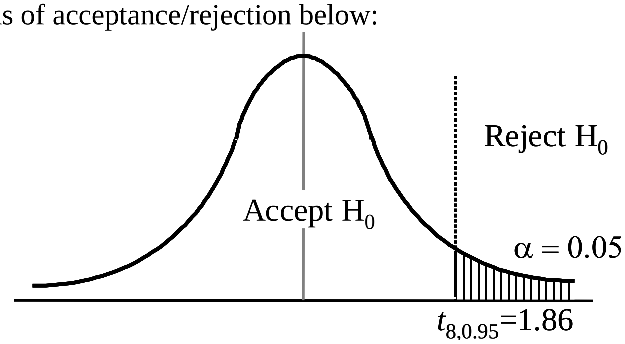
H_1: $\mu > 1500$

Step two: Choose $\alpha = 0.05$. An α of 0.01 is usually reserved for safety of flight questions. At other times, it may be specified in the contract. This is a one tailed test, since we are only going to say the contract was not met if the fuel used is on the high side.

Step three: Since we have less than 30 data points and σ is unknown, use the data to calculate the *t* statistic:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{1600 - 1500}{200/\sqrt{9}} = 1.5$$

Comparing this to the areas of acceptance/rejection below:



Step four: Because $t < t_{\nu,1-\alpha}$ (1.5 < 1.86) we must accept the null hypothesis and accept the contractor's claim that he has met the specification. Another way of saying it is that we don't have the data at 95% confidence to prove that the contractor has failed to meet the specification.

6.6.5 Tests on Variance

The four steps for testing hypotheses on means described in the previous section are still valid here. The only difference in the two procedures is the use here of the chi-squared statistic instead of the *z* or *t* statistic:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

For example, in a bombing system, the mean should be close to zero. Thus, the goodness of a system can best be measured by the dispersion of the system. Generally, the circular error probable is used as a measure of dispersion. We could, however, use the standard deviation.

Suppose the F-19 contract specification states that the standard deviation of miss distances for a particular computed delivery mode shall not exceed 10 meters at the 90% confidence level. In ten test runs, we get a standard deviation of 12 meters. Can we fine the contractor?

Step one: Form null and alternate hypotheses:

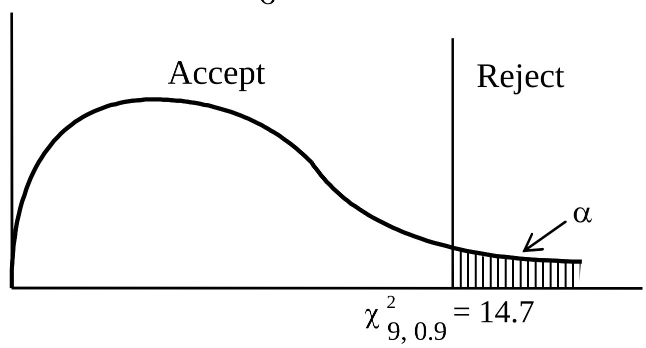
H_0: $\sigma \le 10$

H_1: $\sigma > 10$

Step two: An α of 0.10 is specified. Since smaller σ's are good, our test is a one tailed test. Only extreme large σ's will result in nullifying H_0.

Step three: Using our data, we calculate χ^2 and compare it to $\chi^2_{\nu,1-\alpha}$:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{9(144)}{100} = 13$$



Step four: Because $\chi^2_{9,0.9}$ (13 < 14.7) we do not have adequate data to conclude that the contractor has failed the specification. Accept H_0.

6.6.6 Summary

At times, it can be a little confusing, especially with tests on variances, as to when to reject or accept the null hypotheses. Drawing figures with areas of acceptance and rejection, as has been done in the above examples, can help eliminate the uncertainty. As an aid, the critical regions delineated in Table 6.3 can also be used to define areas of acceptance and rejection.

H_0	Statistics	H_1	Critical Region
$\mu = \mu_0$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ ($n \geq 30$ or σ known)	$\mu < \mu_0$	$z < -z_{1-\alpha}$
		$\mu > \mu_0$	$z > z_{1-\alpha}$
		$\mu \neq \mu_0$	$z < -z_{1-\alpha/2}$ and $z > z_{1-\alpha/2}$
$\mu = \mu_0$	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$ ($n<30$ and σ unknown)	$\mu < \mu_0$	$t < -t_{1-\alpha}$
		$\mu > \mu_0$	$t > t_{1-\alpha}$
		$\mu \neq \mu_0$	$t < -t_{1-\alpha/2}$ and $t > t_{1-\alpha/2}$
$\sigma = \sigma_0$	$\chi^2 = \dfrac{(n-1)s^2}{\sigma^2}$	$\sigma < \sigma_0$	$\chi^2 < \chi^2_{\alpha}$
		$\sigma > \sigma_0$	$\chi^2 > \chi^2_{1-\alpha}$
		$\sigma \neq \sigma_0$	$\chi^2 < \chi^2_{\alpha/2}$ and $\chi^2 > \chi^2_{1-\alpha/2}$

Table 6.3 Test Criteria for Means and Variances

6.7 Nonparametric Tests

The preceding section described tests for populations that have normal or approximately normal distributions. Most phenomena are in fact normal. Some, however, are more accurately described by other distributions, such as the Raleigh, Cauchy, Log Normal, etc. The method of testing hypotheses described is still applicable, but the test statistic and the shape of the probability distribution would change. Tabulated values of these distributions are not always readily available. More frequently, determining the correct distribution type may be difficult. This section describes tests for populations whose distributions are not known to be normal.

6.7.1 Parametric vs Nonparametric Tests

Nonparametric tests make no assumption concerning the shape of the population distribution. These types of tests are less powerful than the tests described in the previous section when they are used on normally distributed data. That is, they require larger sample sizes to give us the same information from the test. Because of this, the preferred procedure would be to use various tests (called goodness of fit tests) to determine the population distribution and then to use the appropriate parametric test. Failing this, a nonparametric test could be used.

Three nonparametric tests that can be useful in flight testing will be presented here: rank sum test, sign test, and signed rank test. The underlying basis for each of these tests is the binomial probability distribution described earlier. Essentially, each test starts out assuming that two populations are equivalent [$f_1(x) = f_2(x)$ and thus $\mu_1 = \mu_2$] and calculates statistics from two samples. Based on these test statistics, you can determine the probability of your observations, assuming identical populations. Given that probability, we can decide if our original assumption was correct.

6.7.2 Rank Sum Test

The rank sum test is also known as the *U* test, the Wilcoxon test, and the Mann-Whitney test in various references. This test, along with the other two nonparametric tests described in this section, can be used to test the null hypothesis that two different samples come from identical populations.

The method consists of the following steps:

1. Rank order all of the data from the two samples, noting whether each data point came from sample one or two.
2. Assign rank values to each point, one to the lowest, two to the next, etc. In the event that two or more data points have the same value, give each an average rank. For instance, if the 7th and 8th points are the same, give both a rank of 7.5.
3. Compute the sum of the ranks of each sample (R_1, R_2).

4. Calculate the following U statistic where n_1 and n_2 are sample sizes

$$U_1 = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1 \text{ and } U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2$$

   Note: $U_1 + U_2 = n_1 n_2$ can be used as a math check.
5. Compare the smaller U to the critical values of U listed in the appendix.
6. If $U <$ critical value, reject H_0: $\mu_1 = \mu_2$.

While the procedure may not appear to be very intuitive, its basis is in the binomial distribution. That is, if two samples are taken from identical populations, what is the probability of getting them in a particular rank order?

As an example, consider the following. The detection ranges of two radars under controlled conditions were tested with the following results:

System 1: 9, 10, 11, 14, 15, 16, 20

System 2: 4, 5, 5, 6, 7, 8, 12, 13, 17

Is there a difference between the two systems at 90% confidence? Using the steps described above:

1. Rank order all scores.
2. Assign Rank values.

Score	4	5	5	6	7	8	9	10	11	12	13	14	15	16	17	20
System	2	2	2	2	2	2	1	1	1	2	2	1	1	1	2	1
Rank	1	2.5	2.5	4	5	6	7	8	9	10	11	12	13	14	15	16

3. Compute R_1, R_2

   $R_1 = 7 + 8 + 9 + 12 + 13 + 14 + 16 = 79$

   $R_2 = 1 + 2.5 + 2.5 + 4 + 5 + 6 + 10 + 11 + 15 = 57$
4. Calculate U_1, U_2

$$U_1 = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1 = 7 \cdot 9 + \frac{7(8)}{2} - 79 = 12$$

$$U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2 = 7 \cdot 9 + \frac{9(10)}{2} - 57 = 51$$

5. Compare the smaller U (12 in this case) with critical values for

   $\alpha = 0.10$ $n_1 = 7$ $n_2 = 9$ $U_{cr} = 15$
6. Since $U < U_{cr}$, we can reject the null hypothesis that the two radars have the same performance with 90% confidence.

6.7.3 The Sign Test

The sign test is an even simpler nonparametric test which has the advantage that it can be applied to ordinal data. All that is required is paired observations of two samples with a "better than" evaluation. For example, this test can be used when each of a group of pilots evaluates two systems and identifies which system each prefers.

Like the rank sum test, the null hypothesis is that the two samples came from the same population and therefore the chance of preferring System A over B is just the same as preferring B over A (i.e., 0.5). Therefore, here we can use the binomial distribution directly. If System A is preferred x times in N tests, the probability of this happening is:

$$f(x) = \frac{N!}{x!(N-x)!} p^x q^{N-x} = \frac{N!}{x!(N-x)!} (0.5)^N$$

(Note that values for $f(x)$ are tabulated in the appendix.) But this is a point probability in our discrete distribution, and we need the entire tail. See Figure 6.1.

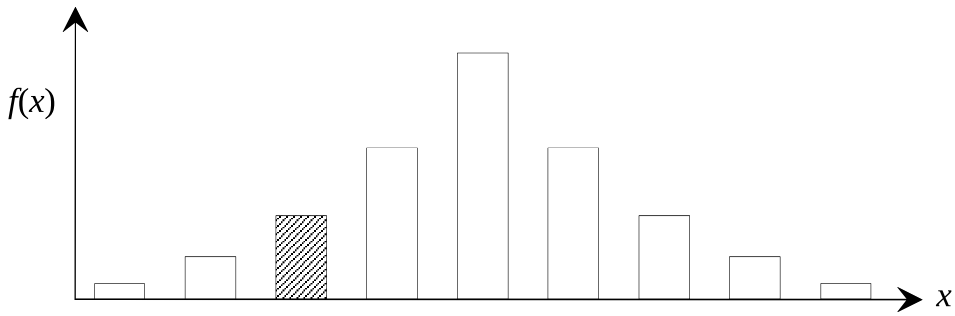


Figure 6.1 Point Probability (shaded area) on a Binomial Distribution

Thus, if you need to test a single tailed hypothesis, then sum the probabilities from the end up to the sample data result:

$$P(0 \le n \le x) = \sum_{i=0}^{x} \frac{N!}{i!(N-i)!}(0.5)^N$$

If the probability of getting a value in the tail(s) of concern is less than your chosen level of significance, then you should reject the null hypothesis that there is no difference between the systems.

For example, suppose 10 pilots evaluate the power approach handling qualities of the F-19 with two different control laws and 7 prefer System B, 2 prefer System A, and 1 has no preference. Should we switch production lines to System B? The cost is high, but if we wait to do more testing the cost will be prohibitive. The null hypothesis is that System A and B are equally desirable. Most math textbooks would have you discard the no-preference data because a binomial system only has two possible results, A ***or*** B in this case. But for a problem like this it may be best to put a no-preference vote in the same category as a vote for the status quo, or null hypothesis. That is, no difference between the two choices. Thus these results should be treated as 7 votes for the new system B, and 3 votes for either A or no preference. Choose a level of significance of 0.10 since safety of flight is not a concern. We must now calculate the probability of getting 0, 1, 2 or 3 pilots to choose system A (or show no preference) even when there really is no significant difference between A and B. If this probability is less than our level of significance, then we will reject H_0 and conclude that B is better than A.

$$P(0 \text{ prefer } A) = \frac{10!}{0!(10!)}\left(0.5^{10}\right) = 0.001$$

$$P(1 \text{ prefer } A) = \frac{10!}{1!(9!)}\left(0.5^{10}\right) = 0.010$$

$$P(2 \text{ prefer } A) = \frac{10!}{2!(8!)}\left(0.5^{10}\right) = 0.044$$

$$P(3 \text{ prefer } A) = \frac{10!}{3!(7!)}\left(0.5^{10}\right) = 0.117$$

$$\text{Total} = 0.172$$

Thus, we can only be 83% sure that B is really better than A. Not enough (at 90% significance) to justify the added expense of System A. Therefore, accept H_0: no significant difference between A and B.

For sample sizes of 15 or larger, we can use the normal approximation to the binomial distribution with very little error. In this case,

$$z \cong \frac{x - np}{\sqrt{npq}}$$

As a comparison, checking the approximation for our last example with n = 10, p = 0.5, q = 0.5, and with x = 3, we get

$$z \cong \frac{3-10(0.5)}{\sqrt{10/4}} = -1.27$$

From the tables in the appendix, this corresponds to single tail probability of 10.2%, 7% off our binomial calculation. The accuracy of the approximation is unacceptable for a sample size of 10, but it improves with larger *n*'s and when $n > 15$, any difference can be neglected.

6.7.4 Signed Rank Test

The signed rank test combines elements of both the sign test and the rank sum test. Thus, the underlying assumptions are the same: system A is no better or worse than System B. Although the sign test was very simple, if we have some indication of how much better System B is than System A, then use of the sign test alone would ignore perhaps crucial data. The signed rank test incorporates this data.

The method is as follows:

1. First, rank the differences between paired observations by absolute magnitude. Ignore cases where a pair of observations is identical (i.e., no preference). Also, if there is a tie in rank order, assign an average rank to each tie.
2. Next, sum the positive and negative ranks (W_+, W_-). The test statistic in the smaller W.
3. Compare W with critical values in the table in the appendix for the appropriate level of significance.
4. Reject H_0 if $W < W_{cr}$

As an example, suppose our previous 10 pilots who evaluated the F-19 flight control system gave systems A and B the following Cooper-Harper ratings (1 best 10 worst) as shown below. Note: implicit in using Cooper-Harper ratings this way is that they contain interval data for this particular test.

Pilot	System A	System B	Difference
1	3	1	2
2	5	2	3
3	3	4	–1
4	4	3	1
5	3	3	0
6	4	2	2
7	4	1	3
8	2	1	1
9	3	1	2
10	1	2	–1

Ranking the differences by absolute magnitude, ignoring the zero difference gives:

Rank	2.5	2.5	2.5	2.5	6	6	6	8.5	8.5
Difference	–1	1	1	–1	2	2	2	3	3

Summing the positive and negative ranks:

$$W_+ = 2.5 + 2.5 + 6 + 6 + 6 + 8.5 + 8.5 = 40.0$$

$$W_- = 2.5 + 2.5 = 5.0$$

Using $\alpha = 0.05$, WCR from the tables in the appendix is 8 (using the one-tailed criteria). Since $W < W_{cr}$ ($5 < 8$), we can now reject H0 and conclude that there is a difference between A and B with 95% confidence.

6.8 Testing for Normality

In the preceding section tests were presented that did not assume the underlying data distribution was necessarily normal. But if the reason for using one of these nonparametric tests is that the experimenter is unsure of the form of the distribution of his or her data, it would be more powerful to test the data distribution and then use a parametric test, if appropriate. This section will present a method for testing a set of data to see if it is likely to be from a normally distributed population.

6.8.1 Probability Plots

The first potential step in testing a data set for a particular distribution could be qualitative. A plot of the data (ordered from low to high) versus the corresponding percentile value of the distribution form will provide a graph that can be visually inspected to determine if the data corresponds to the distribution. If the correspondence is good, the plot will generate a nearly straight line. For a normal distribution assumed, the slope of the straight line curve fit will be σ and the intercept in the y axis will be μ.

To construct a probability plot:

1. order the sample data points from low to high
2. the i^{th} data element is equated to the $[100(i - 0.5)/n]^{th}$ percentile
3. from the assumed distribution table, determine the value for the same percentile
4. pairing the i^{th} data element with the distribution parameter for the same percentile, plot each pair of values on an x-y grid

As an example, consider the following along track (long/short) bomb miss data:

–100	–45	–10	0	40
95	100	0	0	–40

Following the first three steps above, looking at z values from the standard normal distribution the following table is obtained:

i	**data**	**percentile**	**z**
1	–100	5	–1.645
2	–45	15	–1.037
3	–40	25	–0.675
4	–10	35	–0.385
5	0	45	–0.126
6	0	55	0.126
7	0	65	0.385
8	40	75	0.675
9	95	85	1.037
10	100	95	1.645

Finally, taking step four, we plot the data versus the z values shown in the table, obtaining our probability plot:

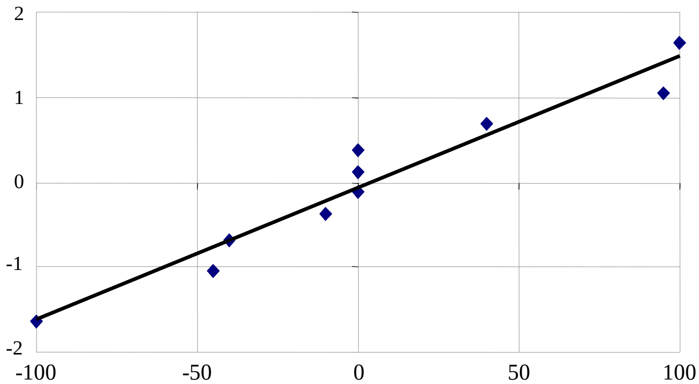


Figure 6.1 Probability Plot

As can be seen from the probability plot in Figure 6.1, the data conforms very closely to the straight line curve fit, qualitatively indicating that the data is likely to be normally distributed.

6.8.2 Correlation Coefficient

For the probability plots in the previous section we had to qualitatively judge whether or not the points fell adequately close to a straight line in order to say if the data conformed to a particular distribution. Better would be a quantitative measure of the "goodness of fit". One possible correlation coefficient is the parameter s_{xy} defined as follows:

$$s_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y})$$

When x_i is large, if y_i is also large, then the contribution to s_{xy} will be positive. And when x_i and y_i are small, resulting in negative deviations, the contribution to s_{xy} will again be positive. Thus large positive values of s_{xy} will indicate good correlation of x and y values. One problem with judging the goodness of fit with such a parameter is that the magnitude of s_{xy} depends on the dimensions of x and y. Since s_{xy} is the sum of the products of the i^{th} deviations, we can non-dimensionalize it by dividing by the product of the square root of the sum of the squared deviations. Calling this r:

$$r = \frac{s_{xy}}{\sqrt{\sum (x_i - \bar{x})^2} \sqrt{\sum (y_i - \bar{y})^2}}$$

An equivalent, but more convenient form for calculating r is:

$$r = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n \sum x_i^2 - \left(\sum x_i\right)^2} \sqrt{n \sum y_i^2 - \left(\sum y_i\right)^2}}$$

R now gives us a quantitative measure of how well our data fits a straight line and hence, how closely the data matches a particular distribution. *R*'s properties include:

1. r is not dependent on units (non-dimensional)
2. r varies between –1 and +1
3. if r =1 then all points would fall exactly on a straight line with positive slope
4. if r = –1 then all points would fall exactly on a straight line with negative slope

As an example, let's calculate r for the data presented in the previous section calculated against an assumed normal distribution. The x values are the data points. The y values are determined from the z values corresponding to the assumed interval as shown in the previous section. The number of data points (n) was 10.

x	y	xy	x^2	y^2
–100	–1.645	164.5	10,000	2.71
–45	–1.037	46.7	2,025	1.08

	–40	–0.675	27.0	1,600	0.46
	–10	–0.385	3.9	100	0.15
	0	–0.126	0	0	0.02
	0	0.126	0	0	0.02
	0	0.385	0	0	0.15
	40	0.675	27.0	1,600	0.46
	95	1.037	98.5	9,025	1.08
	100	1.645	164.5	10,000	2.71
Sums:	40	0.000	532.1	34,350	8.84

Thus r becomes: $\dfrac{10 \times 532.1 - 40 \times 0}{\sqrt{10 \times 34{,}350 - 40^2}\sqrt{10 \times 8.84 - 0^2}}$ or = 0.97

Which indicates a high level of correlation (+1 being the best possible). This correlation coefficient is also available in curve fitting routines in popular personal computer spreadsheets.

6.8.3 Kolmogorov-Smirnov Test

We're still left with a judgment call using the correlation coefficient, r, in the last section. Is 0.97 close enough? Is 0.89 too small? One quantitative test for normality is the Kolmogorov-Smirnov (referred to as K-S) test. Although a little more involved to calculate, the result can be compared to a table of critical values (appendix 7) that vary with sample size and significance level. Thus at the end of this test you can say that you're xx% confident that the data is normally distributed. The procedure for the K-S test is as follows:

1. order the data from low to high
2. calcuate $\frac{x_i - \bar{x}}{s_x}$
3. determine the area under the normal curve up to a z value of $\frac{x_i - \bar{x}}{s_x}$
4. determine the difference (D) between the value in step 3 and i/n
5. compare the largest D to the critical values in appendix 7
6. reject the hypothesis that the data is normal if the max D is > critical value

As an example let's evaluate the bombing data presented in the beginning of this section using the K-S test:

i	x_i	$x_i - \bar{x}$	$\frac{x_i - \bar{x}}{s_x}$	$F\left(\left(x_i - \bar{x}\right)/s_x\right)$	i/n	$i/n - F\left(\left(x_i - \bar{x}\right)/s_x\right)$
1	–100	–104	–1.687	0.046	0.1	0.05
2	–45	–49	–0.795	0.213	0.2	–0.01
3	–40	–44	–0.714	0.238	0.3	0.06
4	–10	–14	–0.227	0.410	0.4	-0.01
5	0	–4	–0.065	0.474	0.5	0.03
6	0	–4	–0.065	0.474	0.6	0.13
7	0	–4	–0.065	0.474	0.7	0.23
8	40	36	0.584	0.707	0.8	0.09
9	95	91	1.476	0.930	0.9	–0.03
10	100	96	1.558	0.940	1.0	0.06

Comparing the largest result for any of the 10 data points (0.23) to the table of critical K-S values in appendix 7, we find that our maximum D is less than any of the critical values for n = 10. Therefore we cannot reject the hypothesis that the data is normally distributed. If our max *D* had been 0.369, we could have stated that the data was not normal with 90% confidence.

6.9 Sample Size

All of the tests presented so far assume the data has all been collected before analysis began. Because collecting data in flight testing can be very costly in terms of money and time (there are always more things to be tested than resources allow), a scientific method to determine how many data points are needed to get statistically significant results would be very useful. We do not want our results obscured by the random variations experienced during the test. On the other hand, excessive sample sizes would give us little additional information at the expense of delaying a lower priority (but required) test.

Presented below are two approaches for determining sample size: accuracy driven and a general approach for establishing a significant difference between means.

6.9.1 Accuracy Driven

If we are required to determine a population statistic (say the mean takeoff distance) within some accuracy (say 10%), then we can use the concept of a confidence interval to determine the number of samples points we need to take. The confidence interval for the mean (σ known) is:

$$p\left(-z_{1-\alpha/2} < \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} < +z_{1-\alpha/2}\right) = 1-\alpha$$

But $|\bar{x}-\mu|$ is the potential error in measuring μ. Thus, we can write:

$$error = \left(\frac{z_{1-\alpha/2}\sigma}{\sqrt{n}}\right) \text{ or}$$

$$n = \left(\frac{z_{1-\alpha/2}\sigma}{Error}\right)^2$$

For example, suppose a review of similar aircraft takeoff data shows that historically the standard deviation is about 20% of the mean. Then if the program office wants us to determine takeoff distance to within 10% with 95% confidence, we can determine the number of times to schedule a takeoff test:

$$z_{0.975} = 1.96, \quad \sigma = 0.2\mu, \quad Error = \pm 0.1\,\mu$$

so

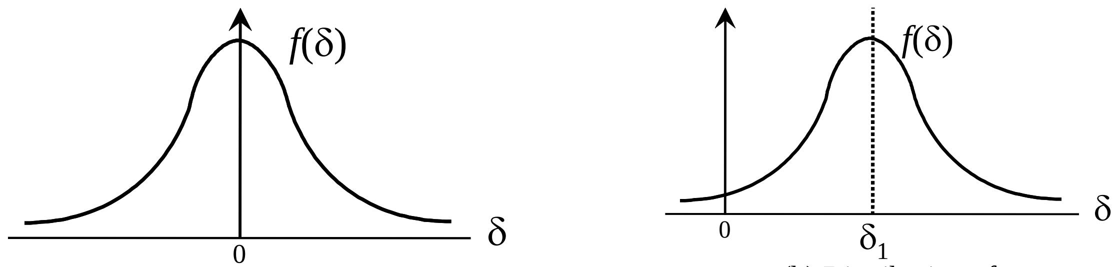
$$n = \left(\frac{(1.96)(0.2\mu)}{0.1\mu}\right)^2 = 15.4$$

Therefore, 16 sorties should be adequate to achieve the accuracy required by the program office. As the test is in progress, we should continually check to see if our assumption concerning the standard deviation remains reasonable (tests of hypotheses on variance).

6.9.2 General Approach

Another frequent problem in flight testing is to determine if a system meets a specification (does $\mu = \mu_0$?) or comparing two systems to see if there is a difference (does $\mu_1 = \mu_0$?). Determining the required sample size is a lot more complex than when the criteria is simply accuracy.

Suppose we sample two different populations with means μ_1 and μ_2. As we take paired samples, we calculate the differences between them, δ. If we took a large number of samples, the resulting δ's would have some mean and distribution. If there really were no difference between the two populations, then the mean would be zero as shown in Figure 6.1(a). If the means were different, then the mean would be some value δ_1 as shown in Figure 6.1(b).



(a) Distribution of $x_1 - x_2 = \delta$ when $\mu_1 = \mu_2$

(b) Distribution of $x_1 - x_2 = \delta$ when $\mu_1 - \mu_2 = \delta$

Figure 6.1

Combining these two alternatives in Figure 6.2, we can see that the two curves cross at some value $\delta = x_C$. A test result that gave a mean of differences above x_C would lead us to conclude that populations one and two differed in their means with level of significance of α. On the other hand, a value less than x_C. would lead us to believe there was not a difference when in fact there was (with probability β as shown). The relationship between α and β can be seen graphically in Figure 6.2. If we move x_C to the right, we reduce α but increase β. Conversely, minimizing β by moving x_C left results in an increase in α. The only way to decrease α and β at the same time is to increase the sample size.

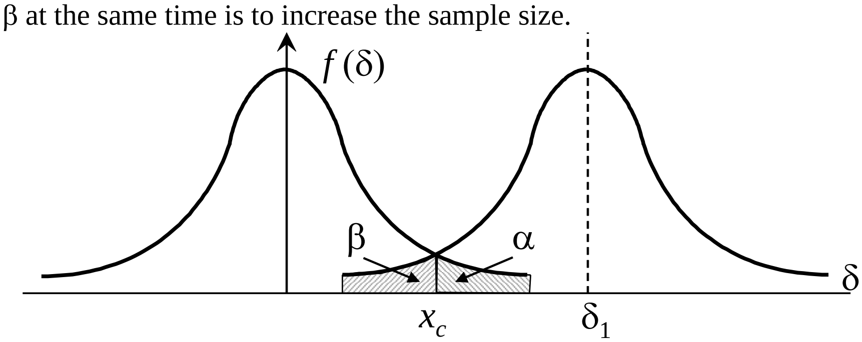


Figure 6.2 Probability of Type I and II Errors for Comparing Means

Recalling from the central limit theorem that $\sigma_{\bar{x}} = \sigma_x / \sqrt{n}$, we can see that α and β are direct functions of the number of samples taken and the value of δ_1. The relationship between these variables is:

$$n = \frac{\left(z_{1-\alpha} + z_{1-\beta}\right)^2 \left(\sigma_1^2 + \sigma_2^2\right)}{\delta_1^2}$$

The way to use this relationship is as follows:

1. Specify α. Normally, 0.10, 0.05, or 0.01.
2. Specify β. Usually larger than α, typically set at 0.10 or 0.20.
3. Specify δ_1. This is the least difference between μ_1 and μ_2 considered operationally significant.
4. Calculate σ_1 and σ_2. Initially, this will come from historical examples or be simply a guess.) As the test continues, it can be refined. Note that if μ_2 is a specification, then μ_2=0.

For example, how many tests are required to determine if the contractor met the specification for a weapon delivery accuracy of 5 mils? Assume a normal error distribution with a standard deviation of 3 mils (from previous tests).

1. Set $\alpha = 0.05$
2. Set $\beta = 0.10$
3. Let $\delta_1 = 1$ mil (operationally significant)

4. σ_1 = 3 mils and σ_2 = 0 (specification)

Now, we can calculate *n*:

$$n = \frac{(Z_{0.95} + Z_{0.90})^2 (3^2 + 0^2)}{1^2} = (1.645 + 1.28)^2 \times 9 = 77$$

Thus, 77 data points are required. Practically speaking, this may be an unacceptable answer, requiring that something in 1, 2, or 3 above be changed. Tradeoffs are the subject of the next section.

6.9.3 Tradeoffs

As can be seen from the example above, we cannot always live with our answers. In calculating *n*, there were many choices, some for which the consequences were not obvious. How significant is it if we change β from 0.10 to 0.20, or if we change δ_1 from 1.0 to 1.5? One good way to approach these choices is to plot the required *n* for various changes in α, β, and δ_1. Then engineering judgment can be used where discretion is available. Figure 6.3 is one such example.

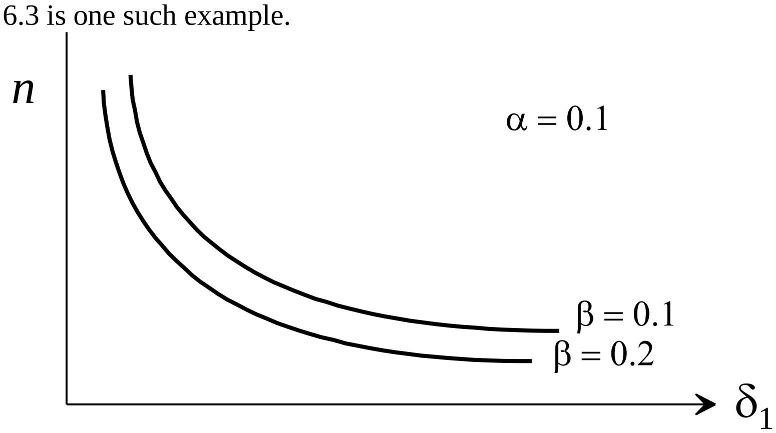


Figure 6.3 Typical Variation of Sample Size, *n*, with Minimum Significant Difference, δ_1

6.9.4 Nonparametric Tests

The required sample size for nonparametric tests cannot be determined with accuracy. In practice, however, it has been found that the signed rank test is about 90% as efficient as a test on means using the *z* statistic. Therefore, you could calculate n as described earlier and divide by 0.9.

For example, how many pilots do we need to evaluate new power approach control laws in the F-19? We want to be 90% certain that there is a significant improvement (defined here as one Cooper-Harper rating).

1. $\alpha = 0.10$
2. $\beta = 0.20$ (arbitrary)
3. $\delta_1 = 1$
4. σ_1, σ_2 (review of similar tests show $\sigma \cong 1$)

Thus,

$$n = \frac{1}{.9} \frac{(Z_{0.90} + Z_{0.80})^2 (1^2 + 1^2)}{1^2} = \frac{1}{9}(1.28 + 0.84)^2 \times 2 = 9.99$$

or 10 evaluation pilots should be planned.

6.10 Circular Error Probable

The use of circular error probable (CEP) is a common statistical method for dealing with errors in two dimensional problems such as navigation systems (along track and across track errors) and in bombing or targeting systems (long/short and left/right errors). A CEP is an error budget that would contain within it one half of all expected errors in the entire population. Another measure commonly used is circular error average (CEA). The difference between the two is that CEP uses the dispersion of the data (standard deviation in each dimension) to predict the population average errors, and CEA is simply an average of the radial miss distance of the sample data. CEA is more easily computed, but does not have the predictive value of a CEP that includes information on the standard deviation of the sample data.

6.10.1 Bivariate Distributions

Since we are dealing with errors in two dimensions (*x* and *y*) the form of the equation that defines the probability distribution function will also have two dimensions. This greatly complicates solution of the distribution function. Even in one dimension for the normal distribution, we had to use a substitution of variables in order to generate a table for the area under the curve (probability). Now with two dimensions, and hence an infinite number of combinations of the standard deviations in *x* and *y*, the problem is increased. The exact equation for CEP (50% probability) of a two dimensional, normally distributed population is as follows:

$$0.5 = \frac{1}{2\pi S_x S_y} \int_0^x \int_0^{\sqrt{CEP^2 - x^2}} e^{\frac{-1}{2}\left[\frac{(x-\bar{x})^2}{S_x^2} + \frac{(y-\bar{y})^2}{S_y^2}\right]} dxdy$$

6.10.2 CEP Approximations

The equation above is based on the bivariate normal distribution and provides accurate results, but requires considerable computing power and must be redone every time you obtain another data point. Most flight testers use approximations for the double integral and these have been shown to be sufficiently accurate for practical use. There are nearly as many approximations as there are as there are authors. The approximation used in the course is the appropriate one of the following:

1. If $S_x < S_y$ and $S_x/S_y < 0.5$ then: CEP = $0.617\ S_x + 0.562\ S_y$
2. If $S_x > S_y$ and $S_y/S_x < 0.5$ then: CEP = $0.562\ S_x + 0.617\ S_y$
3. If neither 1 or 2 above applies then: CEP = $0.5887\ (S_x + S_y)$

6.10.3 Example of CEP Calculation

To illustrate the use of the above approximations reconsider the bomb miss distances previously used to which left/right errors have been added:

i	*x*	*y*
1	-100	0
2	-45	20
3	-10	40
4	0	30
5	40	0
6	95	20
7	100	-20
8	0	-30
9	0	-50
10	-40	-20

Computing the mean values of *x* and *y* gives the mean impact point (MIP): $\bar{x} = +4$ and $\bar{y} = -1$. The MIP coordinates are needed to calculate the sample standard deviations in *x* and *y*. Calculation provides: S_x = 61.6 and S_y = 28.8. Referring to the approximation in the previous section we find that $S_x > S_y$ and the ratio of S_x to S_y is greater than 0.28 (0.47); thus the second approximation applies in this case and we find that:

CEP = $0.615\ S_x + 0.562\ S_y$ = 0.615 (61.6) + 0.562 (28.8) = **54.1**

The individual impacts, the mean impact point, and the CEP are graphically shown in Figure 6.1.

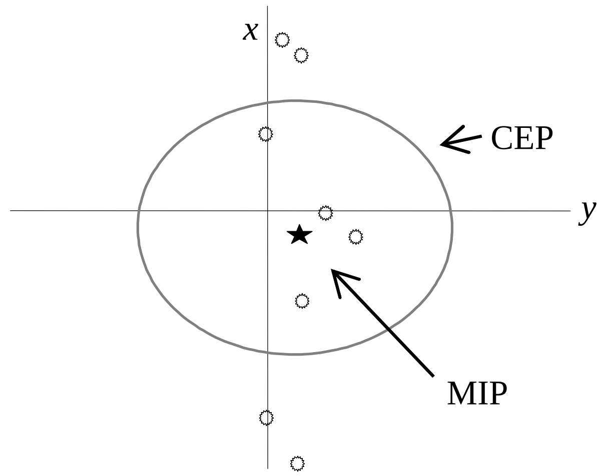


Figure 6.1 Example of CEP

6.10.4 System Bias vs. System Dispersion

As can be seen from the previous figure, our computed CEP is a circle of radius 54.1 centered on the mean impact point. Fifty percent of the entire population of bomb impacts are expected to lie within this circle. But there is a problem with this. Most of the time in practical situations we are concerned with not how well the system groups around a mean point (the MIP), but rather with how closely we can expect the system to hit the target. The difference between the MIP and the target is a measure of system **bias**. The CEP is a measure of system **dispersion**. To meet a specification, both bias and dispersion must be sufficiently small. One way to satisfy this specification concern is to redefine the CEP about the target instead of the MIP. While not strictly correct mathematically we can achieve our objective by redefining the standard deviation as follows:

Normally we define $S_x = \sqrt{\frac{1}{n-1}\sum\left(x_i - \bar{x}\right)^2}$

Redefine S_x as $S_x' = \sqrt{\frac{1}{n-1}\sum\left(x_i\right)^2}$

The effect of this is to arbitrarily define the mean values of *x* and *y* as zero. Put another way, we are assuming that the system has no bias and that the MIP is the target. This will increase the CEP (both s_x and S_y will become larger). From our example in the previous section, recalculating the standard deviations we find:

S_x' = 61.8 (vs. 61.6)
S_y' = 28.9 (vs. 28.8)
CEP′ = 54.2 (vs. 54.1)

Admittedly, there is not much difference in this example. The reason can be seen from examining Figure 6.1. The MIP is very close to the target, as compared to the CEP. Thus, this system has very little bias relative to its dispersion. If there were more bias than dispersion, there would be a greater increase in CEP defined about the target. For contractual objectives it does not matter greatly whether a failure to meet CEP requirements comes from bias or dispersion. So defining the CEP about the target is a satisfactory means to an end, even though it is not mathematically pure.

6.11 Error Analysis

Thus far in the course, we have only been concerned with the statistics of directly measured values. Often, however, measured values are used to compute some parameter of interest. For example, fuel used is usually obtained from fuel flow rate times time $\left(\dot{m} \times t\right)$, and specific range is velocity divided by fuel flow $\left(v/\dot{m}\right)$.

In this section, rules for determining the precision of the computed results are presented. Specifically, we will discuss significant figures, error propagation, and standard deviation of calculated values.

The precision of an experimental result is implied by the way in which the result is written. To indicate the precision, we write a number with as many digits as are significant. The number of significant figures is defined as follows:

6.11.1 Significant Figures

1. The left most nonzero digit is the most significant digit.
2. If there is no decimal point, the rightmost nonzero digit is the least significant digit.
3. If there is a decimal point, the rightmost digit is the least significant digit, even if it is zero.
4. All digits between the least and most significant digits are counted as significant digits.

For example, the following numbers each have four significant digits: 1234, 123,400; 123.4; 1001; 1000.; 10.10; 0.0001010; 100.0.

Although there are no uniform rules for deciding the exact number of digits to use when quoting measured values, the number of significant figures should be approximately one more than that dictated by the experimental precision (i.e., small scale division). For example, if we measure an event using a watch with tenth of a second divisions, we should not record a reading with more than two decimal places (10.24 seconds for instance). When computing a value, the following general rules apply:

1. In addition and subtraction, retain in the more accurate numbers one more decimal digit than is contained in the least accurate number:

   $$1.0 + 3.551 + 4.50 + 1.20 = 1.0 + 3.55 + 4.50 + 1.2 = 10.25$$

2. In all other computations, retain from the beginning one more significant figure in the more accurate numbers than is contained in the least accurate number, then round off the final result to the same number of significant figures as are in the least accurate number:

   $$4.521/2.0 = 4.52/2.0 = 2.26 = 2.3$$

When insignificant digits are dropped from a number, the last digit retained should be rounded off for the best accuracy. To round off a number to a smaller number of significant digits than are specified originally, truncate the number to the desired number of significant digits and treat the excess digits as a decimal fraction. Then

1. If the fraction is greater than 1/2 increment the least significant digit.
2. If the fraction is less than 1/2 do not increment.
3. If the fraction equals 1/2, increment the least significant digit only if it is odd.

Examples: 2.53 = 2.5; 2.56 = 2.6, 2.55 = 2.6, and 2.45 = 2.4

6.11.2 Error Propagation

It should be obvious that the precision of a computed value is dependent on the precision of each directly measured value. In order to show that relationship, consider determining the volume of a right cylinder by measuring the radius and height:

$$V = \pi r^2 h$$

Given that there is some error in each measurement, call them Δr and Δh, producing some error in V, call it ΔV, then

$$V + \Delta V = \pi(r + \Delta r)^2 (h + \Delta h)$$

If the errors in r and h are small, then we can drop products of Δ's after expanding the above equation, as those products will be insignificant in comparison. This gives the following:

$$\Delta V \cong \pi (r^2\Delta h + 2rh\, \Delta r)$$

or $\Delta V \cong \Delta \mathrm{h}(\pi\ r^2) + \Delta r(2\pi\, rh)$

This grouping of the terms reminds one of partial derivatives. Specifically, it is the same as:

$$\Delta V \cong \Delta h\left(\frac{\partial V}{\partial h}\right) + \Delta r\left(\frac{\partial V}{\partial r}\right)$$

In general, it can be shown that for a function Q, where

$$Q = f(a, b, c...)$$

that the error in Q from errors in each independent variable (a, b, c...) is:

$$\Delta Q = \left|\frac{\partial Q}{\partial a}\Delta a\right| + \left|\frac{\partial Q}{\partial b}\Delta b\right| + \left|\frac{\partial Q}{\partial c}\Delta c\right| + \cdots$$

6.11.3 Standard Deviation of a Calculated Value

As we have seen throughout this course, we can't specify the errors in our measurements with certainty. Thus, in the place of the Δ's in the last section, a more usable equation would specify the error in the calculated parameter in terms of the standard deviation of each measured value.
From the definition of variance:

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}(\Delta Q_i)^2$$

Using the earlier approximation for ΔQ,

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial Q}{\partial a}\Delta a_i + \frac{\partial Q}{\partial b}\Delta b_i + \cdots\right)^2$$

Again, dropping cross products as insignificant, we can write

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{\partial Q}{\partial a}\right)^2(\Delta a_i)^2 + \left(\frac{\partial Q}{\partial b}\right)^2(\Delta b_i)^2 + \cdots\right]$$

Since the partial derivatives are common to each summation, they may be taken out:

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial a}\right)^2 \frac{1}{N}\sum_{i=1}^{N}(\Delta a_i)^2 + \left(\frac{\partial Q}{\partial b}\right)^2 \frac{1}{N}\sum_{i=1}^{N}(\Delta b_i)^2 + \cdots$$

where now the term following each partial derivation should be recognized as the definition of variance:

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial a}\right)^2\sigma_a^2 + \left(\frac{\partial Q}{\partial b}\right)^2\sigma_b^2 + \cdots$$

As an example, consider the problem of calculating lift coefficient from the following flight test relationship:

$$C_L = \frac{841.5nW}{V_e^2 S}$$

Assume that the error in S is insignificant in comparison to other errors. What is the standard deviation of C_L for 1% standard deviation in each of n, W, and V_e? First, write

or $$\sigma_{C_L}^2 = \left(\frac{\partial C_L}{\partial n}\right)^2\sigma_n^2 + \left(\frac{\partial C_L}{\partial W}\right)^2\sigma_W^2 + \left(\frac{\partial C_L}{\partial V_e}\right)^2\sigma_{V_e}^2$$

or $$\sigma_{C_L}^2 = \left(\frac{841.5W}{V_e^2 S}\right)^2(0.01n)^2 + \left(\frac{841.5n}{V_e^2 S}\right)^2(0.01W)^2 + \left(\frac{-2\times 841.5n}{V_e^2 S}\right)^2(0.01V_e)^2$$

giving $$\sigma_{C_L}^2 = (0.01)^2 C_L^2 + (0.01)^2 C_L^2 + (-0.02)^2 C_L^2$$

$$\sigma_{C_L} = 0.024C_L$$

Thus, a 1% error in each term results in a 2.4% error in the final result.

6.12 Data Presentation

This section deals with the display of test data to allow quick analysis, to facilitate comparisons, and to permit easy reference to data. Further, by graphically plotting one variable versus another, we may see a correlation (or perhaps as important, a lack of correlation where we expected one) between the two variables. Data smoothing, extrapolation into regions not tested, and interpolation between measured points are all procedures most easily done with a graphical analysis. Thus, good data plots can be an effective analytic tool for flight testing.

6.12.1 Coordinate Scales

A poor choice of scales for the coordinates, more than any other single factor, will make an otherwise acceptable graph unsatisfactory as a tool. Such being the case, the need for suitability rules is evident. Although none can be given to fit all cases, where the maximum revelation of content of data plotted or the maximum of ease and comfort in the use of the plot as a tool are concerned, certain general rules may be stated. Granted the best selection of graph paper, experience has shown it generally desirable to choose the coordinate scales in accordance with the following rules.

1. The scale for the independent variable should be measured along the so called *x*-axis.
2. The scales should be so chosen that the coordinates of any point on the plot may be determined quickly and easily.
3. The scales should be numbered so that the resultant curve is as extensive as the sheet permits, provided the uncertainties of measurement are not made thereby to correspond to more than one of the smallest divisions.
4. Other things being equal, the variables should be manipulated to give a resultant curve which approaches as nearly as practical a straight line.

Sometimes when the data fit a certain type of equation, a straight line graph can be obtained by plotting the measured variables on other than regular rectangular graph paper more simply than by manipulating the variables. For instance, the coordinates $X = \log x$ and $Y = \log y$ are convenient for plotting curves of the form $y^r = ax^n$. Similarly, semilogarithmic paper is especially useful for the graphical analysis of data that are theoretically related by an equation involving the appearance of one of the varieties in the exponent, of the general form $y = Aa^{Bx}$. The coordinates $y = \log_b Y$ and $X = x$ plot a straight line.

6.12.2 Qualitative Curve Fitting

For this discussion, we assume there are sufficient points to justify drawing a smooth continuous curve to represent the actual variation of the related variables under consideration in the regions between the plotted points. Proficiency in judging the most likely course of a smooth curve through a set of plotted points requires practice. There are several basic principles, however, which help us in this task.

Acquire background on similar type data. A priori estimates of what our test results are likely to look like are usually available. Such information as approximate magnitudes and trends are of primary importance in giving us a hint as to what our plot is likely to look like. The source of these estimates can be obtained from classic theory, contractual specifications, military specifications, flight manuals, etc.

The curve which is fitted to the data should be first order, or at most, a second order polynomial. The only time this principle would not be followed is if you knew that the expected wave form is of higher order, such as the dynamic free response of an aircraft.

The curve should be smooth, with few inflections.

The curve should pass as close as reasonably possible to all of the plotted points.

The curve need not pass through a single point, much less through either of the end points. Very often, they are end points because of limits in the accuracy of the instrument or of the method used. In such cases, less weight should be given to them than to the other points of the plot.

The curve should usually, but not always, contain no inexplicable discontinuities, cusps, or other peculiarities.

When taken in moderate sized groups, about one half of the plotted points of each group should fall on one side of the curve and the other half on the other side.

Using these guidelines and good engineering judgment can produce excellent results. On occasion, however, the latitude allowed here may be enough to span the gap between success and failure. In these cases, a mathematical best fit can be used to reduce arguments over how to fit the curve through that data. The method of least squares is one accepted way of defining a mathematical best fit.

6.12.3 Method of Least Squares

To obtain a definition of best fit, consider Figure 6.1 in which the data points are (x_1,y_1), ..., (x_n,y_n). For a given value of x, say x_1, there will be a difference between the value y_1 and the corresponding value as determined from the curve C. We denote this difference by d_1, which may be positive, negative or zero. Similarly, corresponding to the values x_2, ... x_n we obtain the deviations d_2, ..., d_n.

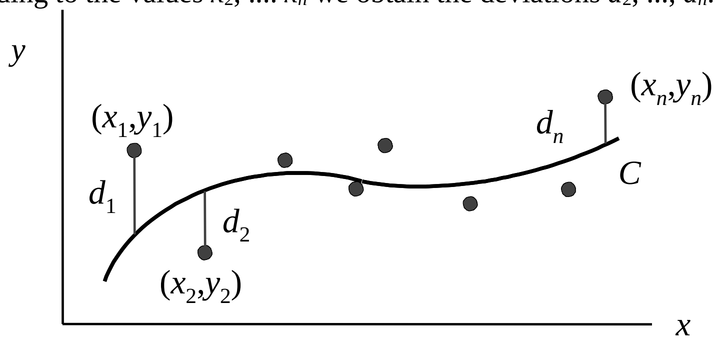


Figure 6.1 Curve Fitting

A measure of the "goodness of fit" of the curve C to the set of data is provided by the quantity Σd_i^2. If this is small, the fit is good; if it is large, the fit is bad. We therefore make the following definition:

Of all curves approximating a given set of data points, the curve having the property that $d_1^2 + d_2^2 + \cdots + d_n^2$ = a minimum is the best fitting curve.

A curve having this property is said to fit the data in the least squares sense and is called a least squares curve. Thus, a line having this property is called a least squares line, a parabola with this property is called a least squares parabola, etc.

For a straight line, for example, the least squares curve can be found as follows. The equation for the line is

$$y = a + bx$$

where a and b must be determined from the available data. For each data point, the deviation, defined above, is:

$$d_i = a + bx_i - y_i$$

and the sum of the squares is:

$$\sum_{i-1}^{N} d_i^2 = \sum_{i-1}^{N} (a + bx_i - y_i)^2$$

To find the minimum sum of the squares, differentiate this expression with respect to both a and b and set the result equal to zero:

$$\frac{\partial}{\partial a}\left(\sum_{i-1}^{N} d_i^2\right) = \sum_{i-1}^{N} 2(a + bx_i - y_i) = 0$$

$$\frac{\partial}{\partial b}\left(\sum_{i-1}^{N} d_i^2\right) = \sum_{i-1}^{N} 2x_i(a + bx_i - y_i) = 0$$

This gives the following two simultaneous equations with two unknowns, a and b.

$$\sum_{i-1}^{N} y_i = an + b\sum_{i-1}^{N} x_i$$

$$\sum_{i-1}^{N} x_i y_i = a\sum_{i-1}^{N} x_i + b\sum_{i-1}^{N} x_i^2$$

For example, if we have the following data:

(x,y) = (1,1), (3,2), (4,4), (6,4), (8,5), (9,7), (11,8), and (14,9)

Then

$\Sigma y = 40$

$\Sigma x = 56$
$\Sigma xy = 364$
$\Sigma x^2 = 524$

Since $n = 8$, we have:

$$40 = 8a + 56b$$
$$364 = 56a + 524b$$

Solving simultaneously, $a = 6/11 = 0.545$, and $b = 7/11 = 0.636$. Thus, the least squares line is:

$$y = 0.545 + 0.626x.$$

Similarly, the least squares parabola which fits a set of sample points is given by:

$$y = a + bx + cx^2$$

where a, b, c are determined from the normal equations

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

While the advantages of the method of least squares are pretty obvious, it does have some disadvantages. Most importantly, all points are given equal weight. Typically in flight testing, the end points are more suspect than middle points. Also, use of the method removes engineering judgment. One approach to the advantages and disadvantages is to use the method first then use engineering judgment to decide if the resulting curve should be used or modified.

6.12.4 Data Rejection

Before concluding, a few remarks are in order about one of the most difficult problems of data analysis: the question of mistakes in the data and the rejection of data.

When the measurement of a quantity is repeated several times, it often happens that one or more of the values differs from the others by relatively large amounts. There is no problem when these anomalous measurements can be directly traced to some systematic disturbance or fluctuation in the controlled conditions of the test. In this case, the values can be corrected for the effects or the data may be rejected. More difficult is the case where no cause for the anomalous values can be ascertained. The analyst is often tempted to discard the anomalous values anyway on the ground that some error in reading the instruments must have occurred. This temptation must be resisted strongly.

The first point to be made is that seemingly large fluctuations are possible, as we have seen in our discussion of distribution error. Thus, it is very often true that the seemingly anomalous values are perfectly acceptable. If the normal probability law indicates that the fluctuation is reasonable, obviously nothing is to be done and the data are certainly to be retained without change.

Now, let us suppose that the deviation we are investigating has a very small chance of occurring. That is, we have computed the chance of obtaining any of our N values with a deviation from the mean as large as was observed, and the probability is calculated to be less than $1/N$. Because of random fluctuations in a series of N measurements, we may reasonably expect very much less. It is a matter of preference at what point one chooses to cut this; a widely used standard is Chauvenet's criterion, which states that if the probability of the value deviating from the mean by the observed amount is $1/(2N)$ or less, the data should be rejected.

For example, consider the following data from repeated measuring of the same parameter:

9, 10, 10, 10, 11, 50

In this case we have six observations and thus we would reject any value if its deviation from the mean has a probability of occurrence less than $1/2n = 1/12 = 0.083$. In normally distributed data that would occur whenever $z \pm 1.73$. The rejection point is the $\pm z$ corresponding to half of the probability because we would reject a point that is either extremely high or low. The value of 50 in our data seems suspect and to estimate its probability of occurrence we use our sample mean (16.7) and standard deviation (16.3) as estimates of the population values to calculate a z for the single data point as follows:

$$z \approx \frac{x-\mu}{\sigma} = \frac{50-16.7}{16.3} = 2.04$$

Because the z corresponding to the data point of 50 is beyond the criteria, we choose to reject the single data point. The new data (only five points now) has a mean of 10 and a much smaller standard deviation of only 0.7. You may apply Chauvenet's criteria only once to each data set.

A distinct danger in applying Chauvenet's or any other criterion for the rejection of data without determinate cause is that important effects may be "swept under the rug." We should rather adopt the view that Chauvenet's criterion should be used to flag suspicious situations. When the deviation observed is larger than one can reasonably expect, this should serve as a stimulus to find out what happened. If it appears that nothing happened, then the data should generally be left as is unless the analyst uses his judgment and experience to determine that it is more likely that the undetected systematic fluctuation occurred than that the effect is real. It cannot be stressed too strongly that judgment is involved here. The blind use of Chauvenet's criterion is a guarantee of never finding anything that was unanticipated at the beginning.

6.13 References

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6.14 Problem Sets

6.14.1 Problem Set 1

1. Your test group has been asked to verify the takeoff performance of an aircraft. Your group decides to do 10 takeoffs all on the same day in the same aircraft without refueling between takeoffs. All 5 pilots want to fly, so it is decided to let each pilot do 2 takeoffs. Are your data:
   a. homogeneous?
   b. independent?
   c. random?

2. Two cards are drawn from a single deck. Find the probability that they are both aces if the first card is:
   a. replaced
   b. not replaced

3. Given the following random, independent 360º aileron roll data:

Test Point	Time to 360 (sec)
1	3.5
2	4.0
3	3.8
4	4.2
5	3.7

   Find the:
   a. Sample Mean
   b. Sample Median
   c. Sample Standard Deviation

6.14.2 Problem Set 2

1. A very large test program was accomplished to determine the stall speed of the F-l9. The standardized data were normally distributed with a mean of 125 knots and a standard deviation of 2 knots. What is the probability of a single stall happening at 130 knots or greater?

2. Assume four random pilots stall the F-19 in problem one. Ninety-nine percent of the time, the sample average stall speed will be less than what value? Assume $s = \sigma$.

3. Ninety-nine percent of the time, the sample standard deviation of the stall speed for the four pilots in problem two will be less than what value?

6.14.3 Problem Set 3

1. You test 50 F-120 engines and find that the mean is 27,900 with a standard deviation of 350 lbs. What are the 90% confidence limits for the actual thrust? (hint: because $n \geq 30$, use z and assume $\sigma = s$)

2. Ten MIL power takeoff rolls were measured by your test group. The standardized data have a mean of 2700 ft and a standard deviation of 200 ft. What are the 95% confidence limits for the actual value?

3. A sample of 9 rocket motors which were stored for 5 years had an average burn time of 3.1 sec and a standard deviation of 0.1 sec. What are the 95% confidence limits for the standard deviation?

6.14.4 Problem Set 4

1. You test 50 F-120 engines and find that the mean thrust is 27,900 lbs with a standard deviation of 350 lbs. The contractual guarantee was 28,000 lbs. Did the contractor meet the guarantee at 90% confidence. (hint: because $n \geq 30$, use z and assume $\sigma = s$)

2. If there were only 11 test engines in problem 1 above, would the contractor have met the guarantee?

3. Rocket motors have a burn time standard deviation of 0.1 sec (σ_0) when produced. A sample of 9 rocket motors which were stored for 5 years have a sample standard deviation of 0.15 sec. At the 95% confidence level, has the standard deviation increased?

6.14.5 Problem Set 5

1. We want to know if the logic in a new RADAR tape has increased detection range. We need to be 95% certain before we give the program office the green light. Given the following data, decide. Do not assume normally distributed data.

Detection Range	
Before:	3, 5, 5, 6, 7, 8, 12, 12, 17
After:	8, 10, 12, 14, 15, 16, 19

2. The YF-l9 has vertical tape instruments. Before going into production, the Program Director polls the test pilots and finds that 8 prefer round dials, 2 have no preference, and 2 want to keep the tapes. You want to be 80% sure before approving a change. What should you do?

3. Ten Weapon System Operators (WSO) have rated a new bomber's two proposed offensive systems stations on a scale of one (best) to five (worst). Assume the data contains interval information. The results are shown below. System A costs 50% more than System B. You want to be 95% confident that System A is significantly better than B. Should we buy System A?

WSO	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
System A	2	1	3	1	2	4	2	3	4	2
System B	3	2	3	3	2	2	4	2	5	3

6.14.6 Problem Set 6

1. Construct a probability plot for the following data using the normal distribution as the assumed distribution.

0	20	40	30	0	20	-20	-30	-50	-20

2. Calculate the correlation coefficient, r.

3. Use the K-S test to test the null hypothesis that the data is normally distributed (use $\alpha = 0.10$).

6.14.7 Problem Set 7

1. Assume that a test will produce normally distributed data with a standard deviation of 5% of the mean ($\sigma = 0.05\ \mu$). How many samples (n) do we need to determine the mean at the 95% confidence level if we want the error to be:

    a. less than 0.10 μ?

    b. less than 0.05 μ?

    c. less than 0.01 μ?

2. A new RADAR component is being tested to determine its effect on detection range. From previous tests, the standard deviation of such tests is about 1.5 nm. How many test points must we fly with both the old and new component if we want to detect a mean difference of 1 nm at 95% confidence while guarding against the false positive with a probability of 90%?

6.14.8 Problem Set 8

1. Find the CEP about the MIP for the following data.

i	*x*	*y*
1	-100	0
2	-45	20
3	-10	40
4	0	30
5	40	0
6	95	20
7	100	-20
8	0	-30
9	0	-50
10	-40	-20
11	210	-20
12	-190	-75

2. Find the CEP about the target (x = y = 0).

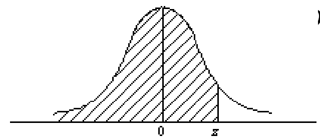
6.14.9 Problem Set 9

1. If the standardized flight test value for specific range (nm per pound of fuel) is 0.082 and the available fuel for cruise is 6,182 lbs, what is the cruise range of the aircraft?

2. Use the method of least squares to find the best straight line to fit the following data:

x	2	7	9	1	5	12
y	13	21	23	14	15	21

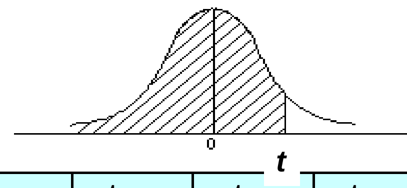
3. If we have six landing distance data points with an average of 3,256 ft and a standard deviation of 175 ft, would you disregard a single data point of 3,954 ft?



6.15 Appendices

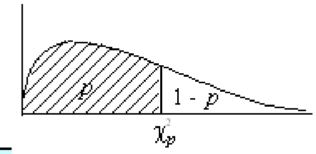
6.15.1 A-1 Standard Normal Distribution

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990



6.15.2 **A-2** Student's *t* Distribution

ν	$t_{0.55}$	$t_{0.60}$	$t_{0.70}$	$t_{0.80}$	$t_{0.90}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$
1	0.158	0.325	0.727	1.376	3.08	6.31	12.71	31.82	63.66
2	0.142	0.289	0.617	1.061	1.89	2.92	4.30	6.96	9.92
3	0.137	0.277	0.584	0.978	1.64	2.35	3.18	4.54	5.84
4	0.134	0.271	0.569	0.941	1.53	2.13	2.78	3.75	4.60
5	0.132	0.267	0.559	0.920	1.48	2.02	2.57	3.36	4.03
6	0.131	0.265	0.553	0.906	1.44	1.94	2.45	3.14	3.71
7	0.130	0.263	0.549	0.896	1.41	1.89	2.36	3.00	3.50
8	0.130	0.262	0.546	0.889	1.40	1.86	2.31	2.90	3.36
9	0.129	0.261	0.543	0.883	1.38	1.83	2.26	2.82	3.25
10	0.129	0.260	0.542	0.879	1.37	1.81	2.23	2.76	3.17
11	0.129	0.260	0.540	0.876	1.36	1.80	2.20	2.72	3.11
12	0.128	0.259	0.539	0.873	1.36	1.78	2.18	2.68	3.05
13	0.128	0.259	0.538	0.870	1.35	1.77	2.16	2.65	3.01
14	0.128	0.258	0.537	0.868	1.35	1.76	2.14	2.62	2.98
15	0.128	0.258	0.536	0.866	1.34	1.75	2.13	2.60	2.95
16	0.128	0.258	0.535	0.865	1.34	1.75	2.12	2.58	2.92
17	0.128	0.257	0.534	0.863	1.33	1.74	2.11	2.57	2.90
18	0.127	0.257	0.534	0.862	1.33	1.73	2.10	2.55	2.88
19	0.127	0.257	0.533	0.861	1.33	1.73	2.09	2.54	2.86
20	0.127	0.257	0.533	0.860	1.33	1.72	2.09	2.53	2.85
21	0.127	0.257	0.532	0.859	1.32	1.72	2.08	2.52	2.83
22	0.127	0.256	0.532	0.858	1.32	1.72	2.07	2.51	2.82
23	0.127	0.256	0.532	0.858	1.32	1.71	2.07	2.50	2.81
24	0.127	0.256	0.531	0.857	1.32	1.71	2.06	2.49	2.80
25	0.127	0.256	0.531	0.856	1.32	1.71	2.06	2.49	2.79
26	0.127	0.256	0.531	0.856	1.31	1.71	2.06	2.48	2.78
27	0.127	0.256	0.531	0.855	1.31	1.70	2.05	2.47	2.77
28	0.127	0.256	0.530	0.855	1.31	1.70	2.05	2.47	2.76
29	0.127	0.256	0.530	0.854	1.31	1.70	2.05	2.46	2.76
30	0.127	0.256	0.530	0.854	1.31	1.70	2.04	2.46	2.75
40	0.126	0.255	0.529	0.851	1.30	1.68	2.02	2.42	2.70
60	0.126	0.254	0.527	0.848	1.30	1.67	2.00	2.39	2.66
120	0.126	0.254	0.526	0.845	1.29	1.66	1.98	2.36	2.62
∞	0.126	0.253	0.524	0.842	1.28	1.64	1.96	2.33	2.58



6.15.3 **A-3** Chi-Square Distribution

ν	$\chi^2_{0.005}$	$\chi^2_{0.01}$	$\chi^2_{0.025}$	$\chi^2_{0.05}$	$\chi^2_{0.10}$	$\chi^2_{0.25}$	$\chi^2_{0.50}$	$\chi^2_{0.75}$	$\chi^2_{0.90}$	$\chi^2_{0.95}$	$\chi^2_{0.975}$	$\chi^2_{0.99}$	$\chi^2_{0.995}$	$\chi^2_{0.999}$
1	0.000	0.000	0.001	0.004	0.016	0.102	0.455	1.32	2.71	3.84	5.02	6.63	7.88	10.8
2	0.010	0.020	0.051	0.103	0.211	0.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6	13.8
3	0.072	0.115	0.216	0.352	0.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8	16.3
4	0.207	0.297	0.484	0.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9	18.5
5	0.412	0.554	0.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7	20.5
6	0.676	0.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5	22.5
7	0.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3	24.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0	26.1
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6	27.9
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2	29.6
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8	31.3
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3	32.9
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8	34.5
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3	36.1
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8	37.7
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3	39.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7	40.8
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2	42.3
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6	43.8
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0	45.3
21	8.03	8.90	10.28	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4	46.8
22	8.64	9.54	10.98	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8	48.3
23	9.26	10.20	11.69	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2	49.7
24	9.89	10.86	12.40	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6	51.2
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9	52.6
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3	54.1
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6	55.5
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0	56.9
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3	58.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7	59.7
40	20.7	22.2	24.4	26.5	29.1	33.7	39.3	45.6	51.8	55.8	59.3	63.7	66.8	73.4
50	28.0	29.7	32.4	34.8	37.7	42.9	49.3	56.3	63.2	67.5	71.4	76.2	79.5	86.7
60	35.5	37.5	40.5	43.2	46.5	52.3	59.3	67.0	74.4	79.1	83.3	88.4	92.0	99.6
70	43.3	45.4	48.8	51.7	55.3	61.7	69.3	77.6	85.5	90.5	95.0	100	104	112
80	51.2	53.5	57.2	60.4	64.3	71.1	79.3	88.1	96.6	102	107	112	116	125
90	59.2	61.8	65.6	69.1	73.3	80.6	89.3	98.6	108	113	118	124	128	137
100	67.3	70.1	74.2	77.9	82.4	90.1	99.3	109	118	124	130	136	140	149

A-4 Critical Values of *U* in the Mann-Whitney Test

α = 0.05 (2 tailed) or α = 0.025 (1 tailed)												
n_1/n_2	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
2	0	0	0	1	1	1	1	1	2	2	2	2
3	2	3	3	4	4	5	5	6	6	7	7	8
4	4	5	6	7	8	9	10	11	11	12	13	13
5	7	8	9	11	12	13	14	15	17	18	19	20
6	10	11	13	14	16	17	19	21	22	24	25	27
7	12	14	16	18	20	22	24	26	28	30	32	34
8	15	17	19	22	24	26	29	31	34	36	38	41
9	17	20	23	26	28	31	34	37	39	42	45	48
10	20	23	26	29	33	36	39	42	45	48	52	55
11	23	26	30	33	37	40	44	47	51	55	58	62
12	26	29	33	37	41	45	49	53	57	61	65	69
13	28	33	37	41	45	50	54	59	63	67	72	76
14	31	36	40	45	50	55	59	64	67	74	78	83
15	34	39	44	49	54	59	64	70	75	80	85	90
16	37	42	47	53	59	64	70	75	81	86	92	98
17	39	45	51	57	63	67	75	81	87	93	99	105
18	42	48	55	61	67	74	80	86	93	99	106	112
19	45	52	58	65	72	78	85	92	99	106	113	119
20	48	55	62	69	76	83	90	98	105	112	119	127

α = 0.10 (2 tailed) or α = 0.05 (1 tailed)												
n_1/n_2	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
2	1	1	1	2	2	2	3	3	3	4	4	4
3	3	4	5	5	6	7	7	8	9	9	10	11
4	6	7	8	9	10	11	12	14	15	16	17	18
5	9	11	12	13	15	16	18	19	22	22	23	25
6	12	14	16	17	19	21	23	25	26	28	30	32
7	15	17	19	21	24	26	28	30	33	35	37	39
8	18	20	23	26	28	31	33	36	39	41	44	47
9	21	24	27	30	33	36	39	42	45	48	51	54
10	24	27	31	34	37	41	44	48	51	55	58	62
11	27	31	34	38	42	46	50	54	57	61	65	69
12	30	34	38	42	47	51	55	60	64	68	72	77
13	33	37	42	47	51	56	61	65	70	75	80	84
14	36	41	46	51	56	61	66	71	77	82	87	92
15	39	44	50	55	61	66	72	77	83	88	94	100
16	42	48	54	60	65	71	77	83	89	95	101	107
17	45	51	57	64	70	77	83	89	96	102	109	115
18	48	55	61	68	75	82	88	95	102	109	116	123
19	51	58	65	72	80	87	94	101	109	116	123	130
20	54	62	69	77	84	92	100	107	115	123	130	138

6.15.4 A-5 Binomial Probabilities for p = q = 0.5

n	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0.250	0.500	0.250										
3	0.125	0.375	0.375	0.125									
4	0.063	0.250	0.375	0.250	0.063								
5	0.031	0.156	0.313	0.313	0.156	0.031							
6	0.016	0.094	0.234	0.313	0.234	0.094	0.016						
7	0.008	0.055	0.164	0.273	0.273	0.164	0.055	0.008					
8	0.004	0.031	0.109	0.219	0.273	0.219	0.109	0.031	0.004				
9	0.002	0.018	0.070	0.164	0.246	0.246	0.164	0.070	0.018	0.002			
10	0.001	0.010	0.044	0.117	0.205	0.246	0.205	0.117	0.044	0.010	0.001		
11	0.000	0.005	0.027	0.081	0.161	0.226	0.226	0.161	0.081	0.027	0.005	0.000	
12	0.000	0.003	0.016	0.054	0.121	0.193	0.226	0.193	0.121	0.054	0.016	0.003	0.000
13	0.000	0.002	0.010	0.035	0.087	0.157	0.209	0.209	0.157	0.087	0.035	0.010	0.002
14	0.000	0.001	0.006	0.022	0.061	0.122	0.183	0.209	0.183	0.122	0.061	0.022	0.006
15	0.000	0.000	0.003	0.014	0.042	0.092	0.153	0.196	0.196	0.153	0.092	0.042	0.014

6.15.5 A-6 Critical Values for the Signed Rank Test

	Two Tailed		One Tailed	
n/α	0.05	0.01	0.05	0.01
5			1	
6	1		2	
7	2		4	0
8	4	0	6	2
9	6	2	8	3
10	8	3	11	5
11	11	5	14	7
12	14	7	17	10
13	17	10	21	13
14	21	13	26	16
15	25	16	30	20
16	30	19	36	24
17	35	23	41	28
18	40	28	47	33
19	46	32	54	38
20	52	37	60	43
21	59	43	68	49
22	66	49	75	56
23	73	55	83	62
24	81	61	92	69
25	90	68	101	77

6.15.6 A-7 Critical Values for the Kolmogorov-Smirnov Test

n/α	0.20	0.15	0.10	0.05	0.01
1	0.900	0.925	0.950	0.975	0.995
2	0.684	0.726	0.776	0.842	0.929
3	0.565	0.597	0.642	0.708	0.828
4	0.494	0.525	0.564	0.624	0.733
5	0.446	0.474	0.510	0.565	0.669
6	0.410	0.436	0.470	0.521	0.618
7	0.381	0.405	0.438	0.486	0.577
8	0.358	0.381	0.411	0.457	0.543
9	0.339	0.360	0.388	0.432	0.514
10	0.322	0.342	0.368	0.410	0.490
11	0.307	0.326	0.352	0.391	0.468
12	0.295	0.313	0.338	0.375	0.450
13	0.284	0.302	0.325	0.361	0.433
14	0.274	0.292	0.314	0.349	0.418
15	0.266	0.283	0.304	0.338	0.404
20	0.231	0.246	0.264	2.940	0.356
25	0.210	0.220	0.240	0.270	0.320
30	0.190	0.200	0.220	0.240	0.290
35	0.180	0.190	0.210	0.230	0.270

Volume 1 – Math & Physics for Flight Testers

Chapter 7

Axis Transformations

Table of Contents

7.1 Introduction

Various axis and coordinate systems have been developed for specific uses. This chapter first introduces these systems, then develops the process for transforming rates and accelerations from one system to another. Understanding this process is required for developing theory for vehicle equations of motion, simulator and inertial navigation system programming. Because transformations are useful in so many fields, they are treated as a separate chapter in this text. To follow this chapter thoroughly, a good understanding of vector analysis and basic matrix algebra including multiplication and inverse operations is a prerequisite.

7.2 Coordinate Systems

There are two primary coordinate systems that are useful in the analysis of vehicle motion; the inertial and the vehicle coordinate systems. Newton's laws apply only when observed from inertial space, but practical instrumentation is strapped to the test vehicle. Before attempting to develop transformation equations, it is first appropriate to define the various inertial and vehicle systems. According to convention, all coordinate systems used will be right-hand orthogonal.

7.2.1 Inertial Coordinate System

An inertial coordinate system is necessary to employ Newton's second law. Any system without acceleration or rotation qualifies as a true inertial system. The problem is that just about any place in the universe does have some acceleration or rotation. The earth may seem fixed, but we know that it is continually spinning and rotating about the sun. It might seem then that a coordinate system placed in the sun would be non-accelerating until we remember that our sun, in fact, the whole galaxy is accelerating within the universe. The only truly inertial reference frame is probably at the center of the universe. Unfortunately, not only do we not know where the center of the universe is, but we also have no way of making any practical measurements of motion relative to it.

The good news is that the influence of the moving galaxy and earth have such a small affect on the outcome of Newton's laws, that within our ability to measure, we'd get the same answer whether using the true inertial reference or some other "sufficient" reference. For example, consider the acceleration A of a billiard ball with mass M after it is hit by a cue ball with force F. From Newton's second law we know the acceleration will be $A = F/M$. This can be verified with extreme accuracy using a reference system that is anchored to the billiard table. While we know that the table is certainly not a true inertial reference frame, the small distances and time of travel render it sufficient.

In contrast, the study of orbital mechanics would be quite flawed if we used the same earth-based reference system. The motion of a satellite is affected by the planet's motion about the sun, so the sun would be the sufficient inertial reference system (Figure 7.1).

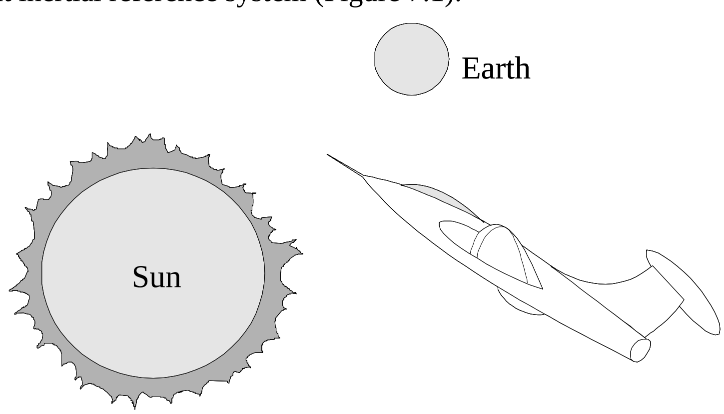


Figure 7.1 The Inertial Coordinate System

The study of vehicle motion due to forces is called the *equations of motion* and is based on Newton's second law. Applying this law requires a sufficiently inertial reference system for the vehicle under question. Experience with aircraft shows that the earth axis system is a sufficient approximation for an inertial coordinate system.

7.2.1.1 Earth Axis System

There are two earth axis systems, the fixed and the moving. Both are referred to with the capital letters XYZ for the three axes. An example of a moving earth axis system is an inertial navigation platform. An example of a fixed-earth axis system is a ground radar site (Figure 7.2).

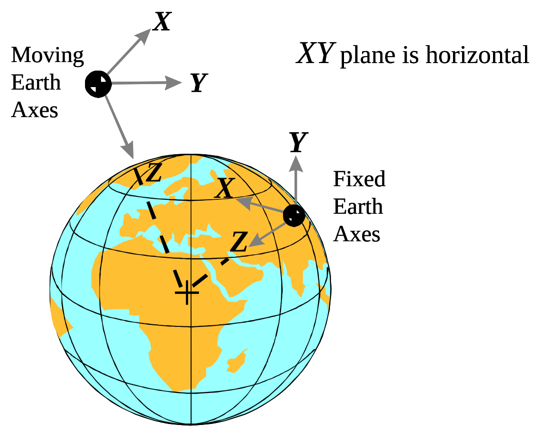


Figure 7.2 The Earth Axis Systems

In both earth axis systems, the *Z*-axis points toward the center of the earth along the gravitational vector, *g*. The *XY*-plane is parallel to the local horizontal while the orientation of the *X*-axis is arbitrarily defined (usually North). The two earth axis systems are distinguished by the location of their origins. The origin of the fixed system is usually taken as some arbitrary location on the earth's surface. The origin of the moving system is usually taken as the aircraft's *cg*. What distinguishes the moving earth axis system from the vehicle axis system is that the moving earth axes are not fixed in orientation with respect to the aircraft. They are instead fixed with respect to local vertical (and North). Throughout this chapter, the *XYZ* (upper case) system will be the fixed-earth axis system unless otherwise noted.

7.2.2 Vehicle Axis Systems

These coordinate systems have origins and axes defined with respect to the aircraft and are fixed to the aircraft once established. The four, which are commonly used for describing aircraft motion, are the body, stability, wind, and principle axis systems. The body and stability axis systems are used for developing the simplified equations of motion. The wind and principle axis systems are not used for basic equations of motion but are used in roll coupling analysis and other advanced topics.

Body Axes

The body axis system is the most straightforward. The axes originate at the aircraft *cg* and extend along the body as shown in Figure 7.3. The positive x-axis points forward along an aircraft horizontal reference line with the positive y-axis out the right wing. The positive z-axis points downward out the bottom of the aircraft, the xz-plane is usually the vehicle plane of symmetry. Note that lower case xyz are used. The unit vectors are $\hat{i}$, $\hat{j}$, and $\hat{k}$ also have origins at the aircraft *cg*.

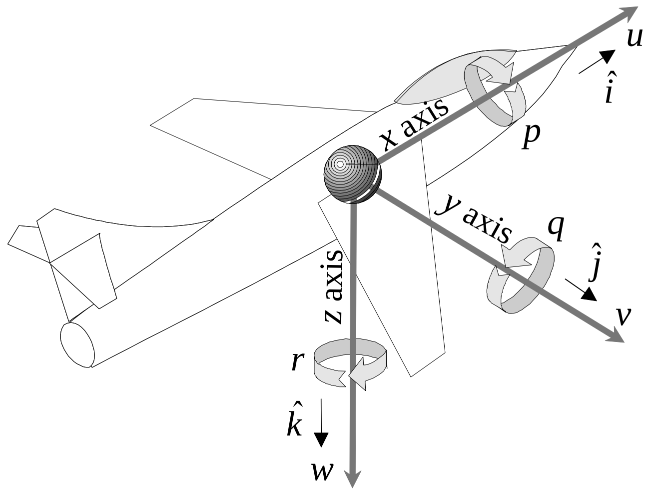


Figure 7.3 Body Axis Systems

Total aircraft linear velocity can be broken down into the three orthogonal components along the body axis system:

u = forward velocity, along the positive x-axis

v = side velocity, along the positive y-axis

w = vertical velocity, along the positive z-axis

Similarly, the total aircraft rotational rates can be broken down into the three orthogonal components about the body axes:

p = roll rate about the x-axis (positive for right roll)

q = pitch rate about the y-axis (positive for pitch up)

r = yaw rate about the z-axis (positive for yaw right)

Application of the body axis system ensures that the moments and products of inertia are constant (assuming constant mass distribution) and that aerodynamic forces and moments depend only upon the relative velocity orientation angles α and β. The body axis system is also the natural frame of reference for most airframe-mounted instrumentation.

7.2.2.1 Stability Axes

Assuming zero sideslip, the stability axis system is just like the body system except that the x_s axis points into the relative wind instead of along the nose. This can be easily visualized by rotating the body system about the y axis by α degrees. Obviously then, the difference between x_b and x_s is α, the difference between z_b and z_s is also α, and the y_b and y_s axes are coincident. The α used in this operation is the equilibrium flight (trim) value. This initial realignment does not alter the body-fixed nature of the axis system. In other words, once established, the stability axis system remains fixed to the body for that application. Of course, each application may have a different value for trim α, and therefore a different moment of inertia and product of inertia along the axes.

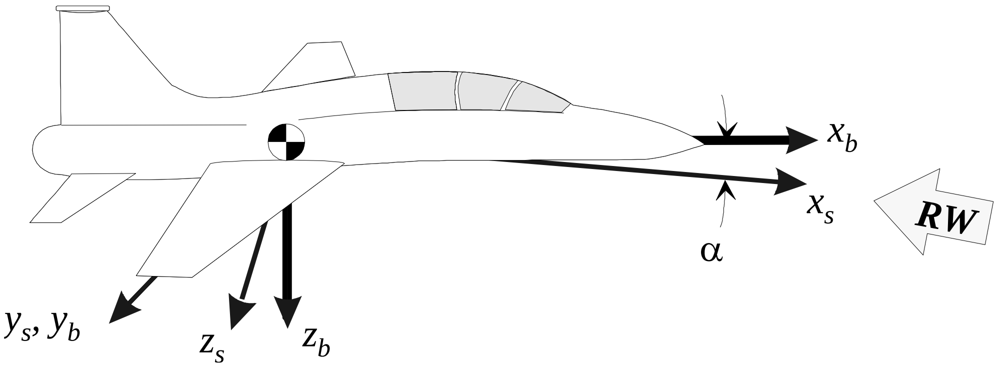


Figure 7.4 Stability Axis System

From the geometry of this figure, any force, velocity or acceleration along any stability axis can be transformed to the body axes as follows:

$$x_b = x_s \cos\alpha - z_s \sin\alpha \qquad z_b = z_s \cos\alpha + x_s \sin\alpha \qquad y_b = y_s$$

7.2.2.2 Wind Axes

Wind axes are oriented with respect to the flight path of the vehicle, i.e., with respect to the relative wind V_T. If the reference flight condition is symmetric, then sideslip is zero and the wind axes coincide with the stability axes. Just as the difference between x_b and x_s illustrates α, the difference between x_s and x_w illustrates β, Figure 7.5a. The equations with the figure show how to transform velocity, force or acceleration along any wind axis to the stability axes. The relationship of true velocity and its components to α and β and the body axis coordinate system is shown in Figure 7.5b.

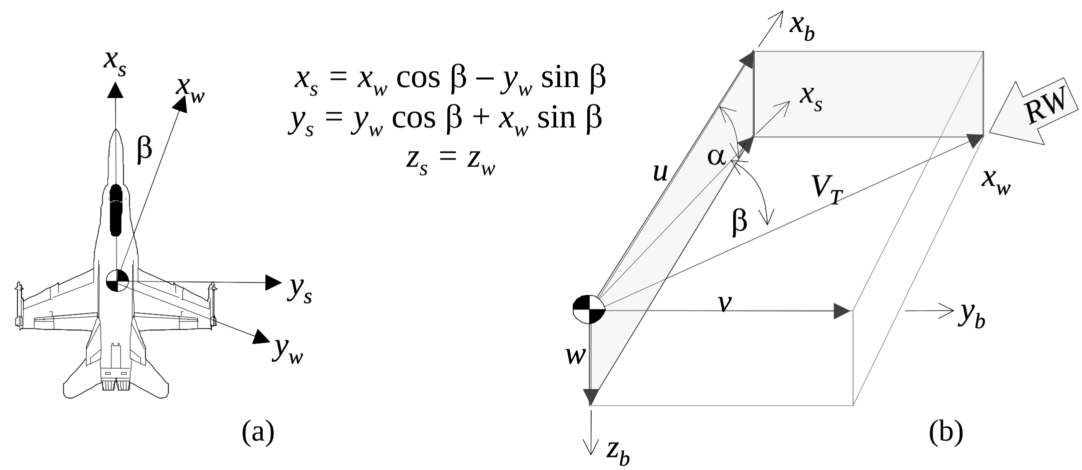


Figure 7.5 Velocity Components and the Aerodynamic Orientation Angles α and β

The complete transformation [of forces, velocity, or accelerations] from the wind axis system to the body system is simply the combination of the previous two transforms. This shown below in matrix form.

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 - \sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_y \\ y_w \\ z_w \end{bmatrix} \tag{7.1}$$

The inverse operation converts any body axis forces, velocities or accelerations to the flight path system

$$\begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \quad (7.2)$$

Using Figure 7.5, angles α and β can be expressed in terms of the velocity components:

$$\sin\alpha = \frac{w}{V_T \cos\beta}$$

ASSUMING β is small, then cos β ≅ 1 [see Table 1.1] and $\alpha = \frac{w}{V_T}$

ASSUMING α is also small, then sin α ≅ α [units in radians, see Table 1.1] and $\alpha = \frac{w}{V_T}$

For angle of sideslip: $\sin\beta = \frac{v}{V_T}$

If β is small, then sin β ≅ β and: $\beta = \frac{v}{V_T}$

β (deg)	**cos β**	**sin β**	**β** (rad)
0°	1	0	0
5°	0.9962	0.0872	0.0872
10°	0.9848	0.1736	0.1745
15°	0.9659	0.2588	0.2618
20°	0.9396	0.3420	0.3491
25°	0.9063	0.4226	0.4363
30°	0.8660	0.5000	0.5236

Table 7.1 Trigonometric Values for Small Angles

7.2.2.3 Principle Axes

Principle axes are the natural axes of rotation of the aircraft when only the mass properties are considered and aerodynamic effects are neglected. The orientation of these axes relative to the aircraft is a function of the mass distribution and are those axes where all of the products of inertia are reduced to zero. This can be seen as aligning the axes with "dumbbells" that duplicate the aircraft's mass distribution along each axis (Figure 7.6).

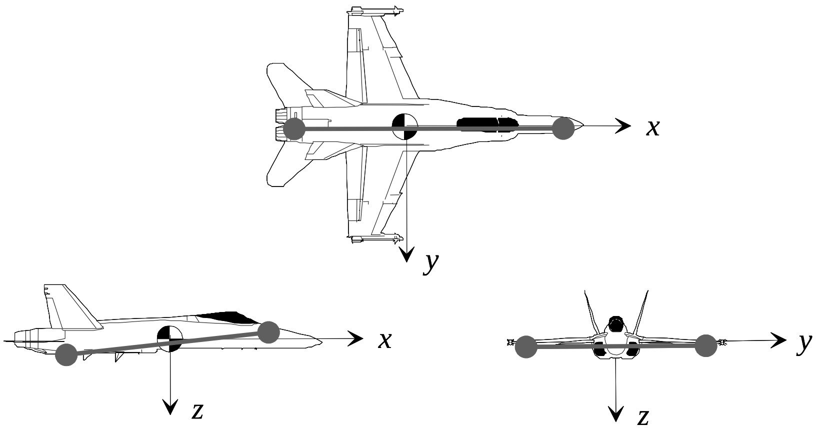


Figure 7.6 Principle Axes are Aligned with Mass Distribution

The moments of inertia are defined as:

$$I_x \equiv \int (y^2 + z^2) dm \quad (7.3)$$

$$I_y \equiv \int (x^2 + z^2) dm \quad (7.4)$$

$$I_z \equiv \int (x^2 + y^2) dm \quad (7.5)$$

These are measures of rotational inertia about their respective axes. The integrations performed in Equation 7.4 can be visualized using Figure 7.7 where the square of the distance for each point of mass is added up.

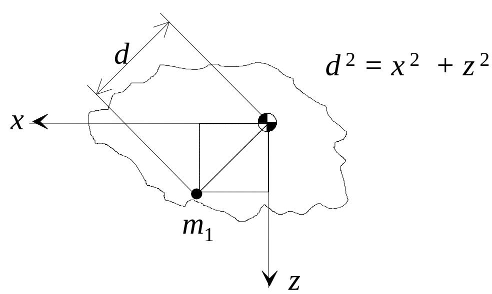


Figure 7.7 Moment of Inertia, I_y

Considering the layout of any aircraft, I_z is always the largest value. $I_y > I_x$ for fuselage-loaded aircraft and $I_x > I_y$ for wing-loaded aircraft.

The products of inertia are defined as:

$$I_{xy} = I_{yx} \equiv \int xydm \quad (7.6)$$

$$I_{yz} = I_{zy} \equiv \int yzdm \quad (7.7)$$

$$I_{xz} = I_{zx} \equiv \int xzdm \quad (7.8)$$

Products of inertia are measures of asymmetry. Figure 7.8a shows an object with its mass distributed about the *x* and *y* "body" axes. Integrating this mass according to Equation 7.6 essentially means concentrating the mass from opposite quadrants into the appropriate size "dumbbells", one positive and one negative as shown in Figure 7.8b. The principle axis is essentially the average of these dumbbells. When they have different tilts or sizes due to asymmetric mass distribution, the principle (average) axis lies along some line different from the body axis. The value for I_{xy} reflects both the magnitude and the tilt of this misalignment. When the mass distribution is symmetric about some line (as shown in Figure 7.6a and 7.6c), the dumbells are symmetrically tilted and massed. In this case, the average weight lies in a line that is coincident with a body axis. The product of inertia is therefore zero for views having a plane of symmetry.

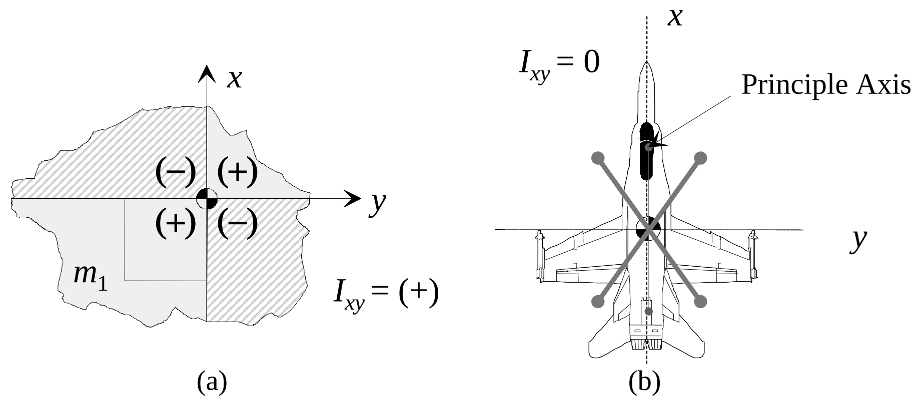


Figure 7.8 Visualizing the Product of Inertia

The inclination of the x_p axis relative to the x_b axis has a direct bearing on the inertial moments experienced about the body axes as reflected by the product of inertia term I_{xz} in the equations of motion. The equations of motion would be simplified if these axes were used, but it is difficult to accurately describe the aircraft motion in this system. Principle axes are not generally used in the *basic* analysis of the motion of an aircraft. They are, however, used in more advanced studies such as roll coupling and spins.

7.3 Euler Angles

The orientation of any coordinate system relative to another can be given by three "Euler" angles, which are consecutive rotations about the *z*, *y*, and *x* axes. They carry one frame into coincidence with another. In flight dynamics, the Euler angles used are those, which rotate the earth axis system into coincidence with the relevant vehicle axis system (Figure 7.9).

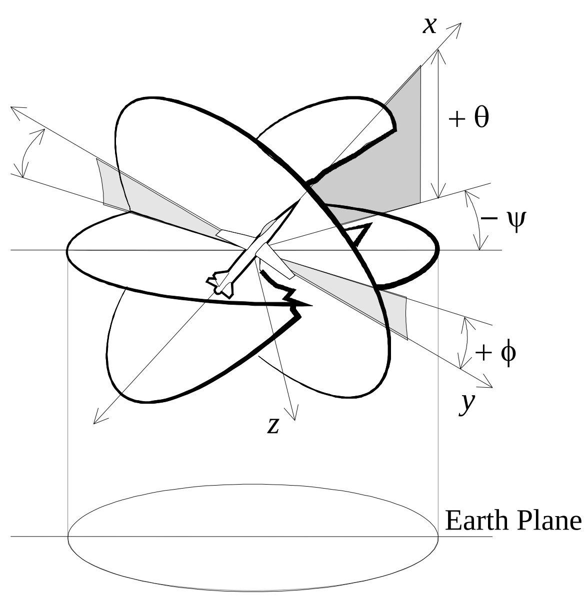


Figure 7 .9 The Euler Angle Rotations

Euler angles are expressed as yaw (ψ), pitch (θ), and roll (ϕ). The sequence: first yaw, then pitch, then roll; must be maintained to arrive at the proper orientation angles. The Euler angles are defined as follows:

ψ- Yaw Angle: The angle between the projection of the vehicle x_b-axis onto the horizontal reference plane and the initial reference position of the earth *x*-axis. Note that yaw angle is the vehicle heading only if the initial reference is north.

θ - Pitch Angle: The angle measured in the vertical plane between the vehicle x_b-axis and the horizontal reference plane.

ϕ - Roll Angle: The angle, measured in the *yz*-plane of the body-axis system, between the *y*-axis and the horizontal reference plane. This is the same as bank angle and is a measure of the rotation (about the *x*-axis) to return the aircraft to a wings level condition.

The importance of the sequence of the Euler angle rotations cannot be overemphasized. Finite angular displacements do not behave as vectors. Therefore, if the sequence is performed in a different order than ψ, θ, ϕ, the final result will be different. This fact is clearly illustrated by the final aircraft attitudes shown in Figure 7.10 in which two rotations of equal magnitude are performed about the *x* and *y* axes, but, in opposite order. Addition of a rotation about a third axis does nothing to improve the outcome.

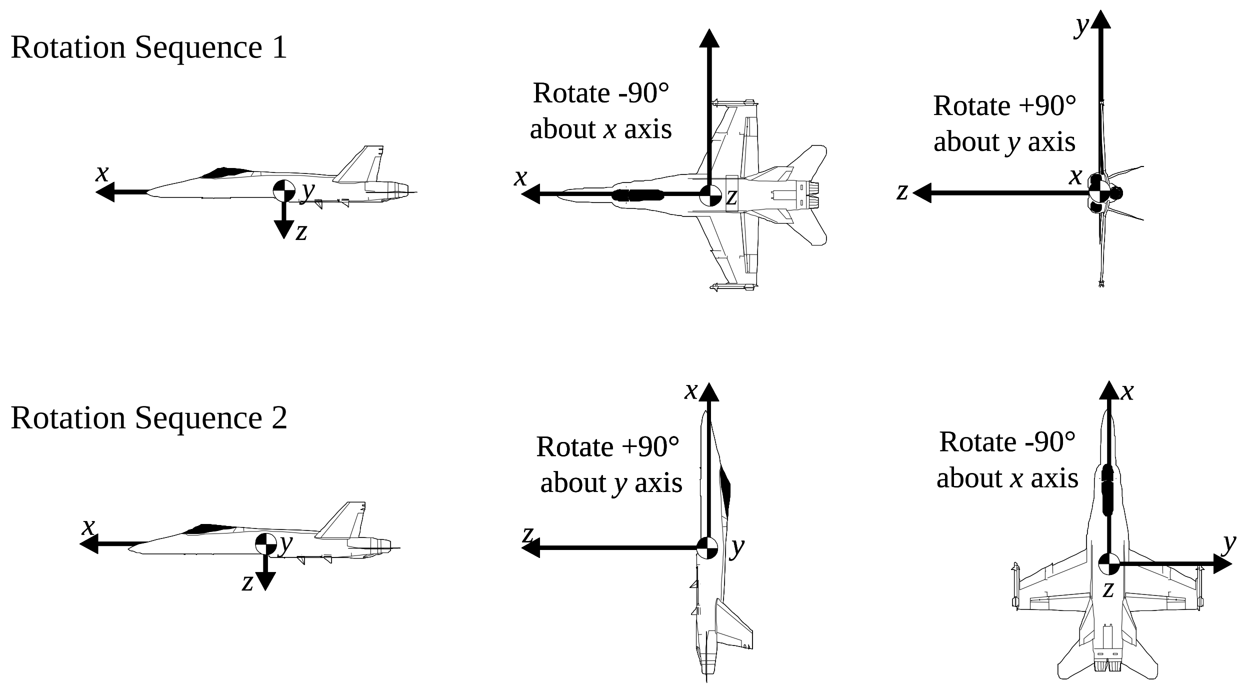


Figure 7 .10 Finite Angular Displacements do Not Behave as Vectors

7.3.1 Force and Velocity Transformations

An Eulerian axis system (*xyz*) is particularly useful in the study of airframe dynamics in that moments and products of inertia measured relative to the Eulerian axes are independent of time for the duration of any particular dynamic analysis and because these axes do not move with respect to the airframe. We can express the location of point *P* in Figure 7.11 by the position vector (r_0):

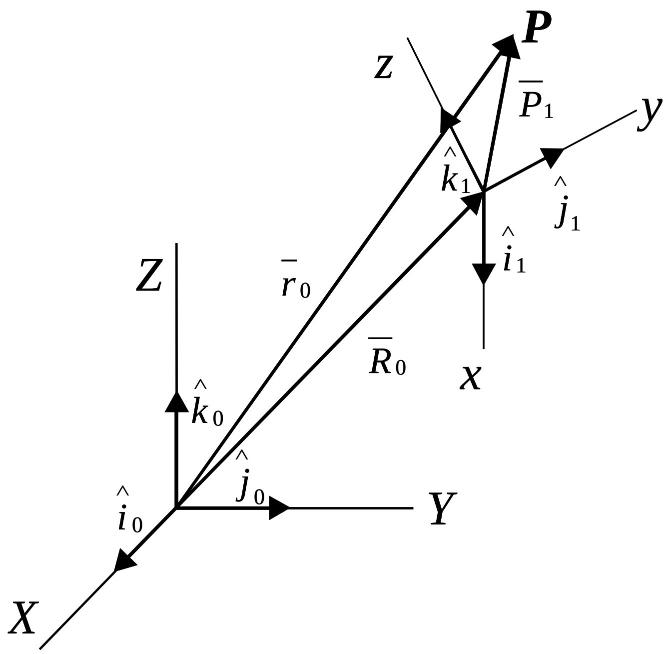


Figure 7 .11 Development of the Transformation Equations

$$\bar{r}_0 = \overline{R}_0 + \overline{P}_1$$

or in component form:

$$x_0\hat{i}_0 + y_0\hat{j}_0 + z_0\hat{k}_0 = X_0\hat{i}_0 + Y_0\hat{j}_0 + Z_0\hat{k}_0 + x_1\hat{i}_1 + y_1\hat{j}_1 + z_1\hat{k}_1 \qquad (7.9)$$

We can determine the components of the above position vector in any direction by forming the dot product of Equation 7.9 with a unit vector in the desired direction, i.e.:

$$x_0 = X_0 + x_1\hat{i}_1 \cdot \hat{i}_0 + y_1\hat{j}_1 \cdot \hat{i}_0 + z_1\hat{k}_1 \cdot \hat{i}_0$$

$$y_0 = Y_0 + x_1\hat{i}_1 \cdot \hat{j}_0 + y_1\hat{j}_1 \cdot \hat{j}_0 + z_1\hat{k}_1 \cdot \hat{j}_0$$

$$z_0 = Z_0 + x_1\hat{i}_1 \cdot \hat{k}_0 + y_1\hat{j}_1 \cdot \hat{k}_0 + z_1\hat{k}_1 \cdot \hat{k}_0$$

where the dot product of the various unit vectors represents the "direction cosines" between the coordinates. Direction cosines are discussed thoroughly in Chapter 2, Inertial Navigation Systems.

A point on a rigid body can be defined in terms of body-fixed (*xyz*) axes and by three independent Euler angles ψ, θ and ϕ defining the angular orientation of the body axes relative to the inertial axes (*XYZ*). To do this, start with the body axes coinciding with the inertial axes at position 0. Then allow the body axes to rotate about the z_0-axis through an angle ψ. The relationship between the two coordinates is then given by:

$$x_1\hat{i}_1 + y_1\hat{j} + z_1\hat{k}_1 = x_0\hat{i}_0 + y_0\hat{j}_0 + z_0\hat{k}_0$$

Employing the dot product concept introduced in Equations 7.9, we get the components with respect to the new axes at position 1 as shown in Figure 7.12:

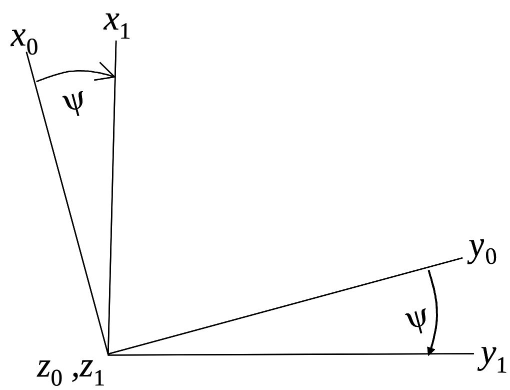


Figure 7 .12 Rotation about z_0 through the Angle ψ

$$x_1 = x_0\hat{i}_0 \cdot \hat{i}_1 + y_0\hat{j}_0 \cdot \hat{i}_1 + z_0\hat{k}_0 \cdot \hat{i}_1$$
$$y_1 = x_0\hat{i}_0 \cdot \hat{j}_1 + y_0\hat{j}_0 \cdot \hat{j}_1 + z_0\hat{k}_0 \cdot \hat{j}_1$$
$$z_1 = x_0\hat{i}_0 \cdot \hat{k}_1 + y_0\hat{j}_0 \cdot \hat{k}_1 + z_0\hat{k}_0 \cdot \hat{k}_1$$

In matrix form, this becomes:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \hat{i}_0 \cdot \hat{i}_1 & \hat{j}_0 \cdot \hat{i}_1 & \hat{k}_0 \cdot \hat{i}_1 \\ \hat{i}_0 \cdot \hat{j}_1 & \hat{j}_0 \cdot \hat{j}_1 & \hat{k}_0 \cdot \hat{j}_1 \\ \hat{i}_0 \cdot \hat{k}_1 & \hat{j}_0 \cdot \hat{k}_1 & \hat{k}_0 \cdot \hat{k}_1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_o \\ z_o \end{bmatrix}$$

For the second rotation, allow the body axes to rotate about the y_1 -axis through an angle θ. The components with respect to the new axes at position 2 are as shown in Figure 7.13:

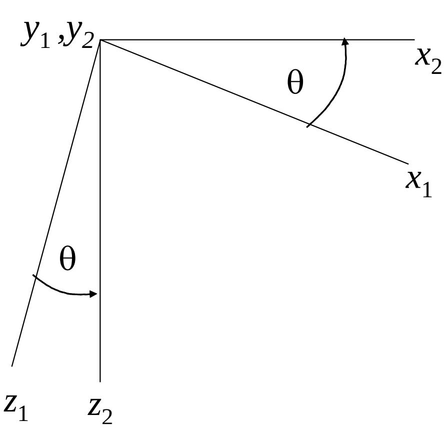


Figure 7 .13 Rotation about y_1 through the Angle θ

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \hat{i}_1 \cdot \hat{i}_2 & \hat{j}_1 \cdot \hat{i}_2 & \hat{k}_1 \cdot \hat{i}_2 \\ \hat{i}_1 \cdot \hat{j}_2 & \hat{j}_1 \cdot \hat{j}_2 & \hat{k}_1 \cdot \hat{j}_2 \\ \hat{i}_1 \cdot \hat{k}_2 & \hat{j}_1 \cdot \hat{k}_2 & \hat{k}_1 \cdot \hat{k}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

Finally, for the last rotation, allow the body axes to rotate about the x_2-axis through an angle ϕ. The components with respect to the new axes at position 3 are as shown in Figure 7.14:

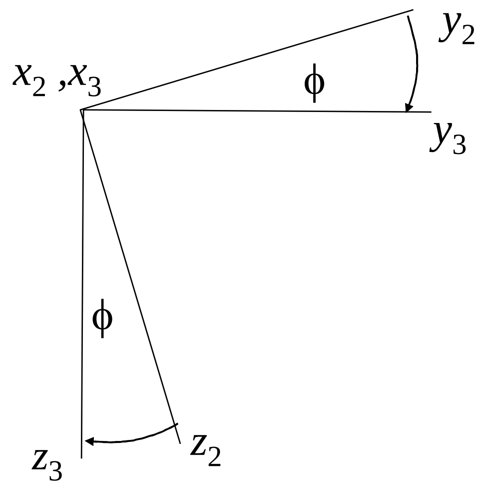


Figure 7 .14 Rotation about x_2 through the Angle ϕ

$$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} \hat{i}_2 \cdot \hat{i}_3 & \hat{j}_2 \cdot \hat{i}_3 & \hat{k}_2 \cdot \hat{i}_3 \\ \hat{i}_2 \cdot \hat{j}_3 & \hat{j}_2 \cdot \hat{j}_3 & \hat{k}_2 \cdot \hat{j}_3 \\ \hat{i}_2 \cdot \hat{k}_3 & \hat{j}_2 \cdot \hat{k}_3 & \hat{k}_2 \cdot \hat{k}_3 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

Again, the order in which these rotations are taken is very important. The position vector will not arrive at the same position if the order of the rotations is changed. The standard order in which the rotations are taken in aircraft dynamic analysis ψ, θ, ϕ.

The combined transfer matrix for converting forces or velocities from the inertial axes to the body axes coordinate system becomes:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (7.10)$$

Expanding, gives:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (7.10a)$$

The inverse of the transform matrix for converting from the body axes to the inertial axes coordinate system can be found with a little effort (by matrix inversion) to be:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ -\sin\phi & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (7.11)$$

7.3.2 Angular Rate Transformations

Frequently, we need to express the angular velocities p, q, and r about the body axes (*xyz*) in terms of their components in the inertial axis system (*XYZ*) and the Euler angles. By the use of the transfer matrix process described above, an angular velocity vector ω, expressed in terms of inertial-axis components, can be transformed to components in a body-oriented axis system. To do this, we must perform the following sequence of steps. With the body axis system coinciding with the inertial axis system, first allow the body axes to rotate about the z_0-axis with an angular velocity of $\dot{\psi}$ (Figure 7 .15):

$$\begin{bmatrix} p_1 \\ q_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

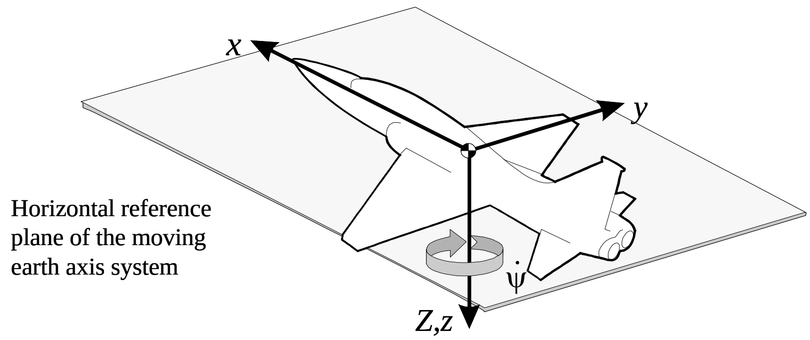


Figure 7.15 Development of Aircraft Angular Velocities by the Euler Angle Yaw Rate (ψ rotation)

The body axis system is then rotated about the y_1-axis with an angular velocity of $\dot{\theta}$ (Figure 7.16):

$$\begin{bmatrix} p_2 \\ q_2 \\ r_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{bmatrix}$$

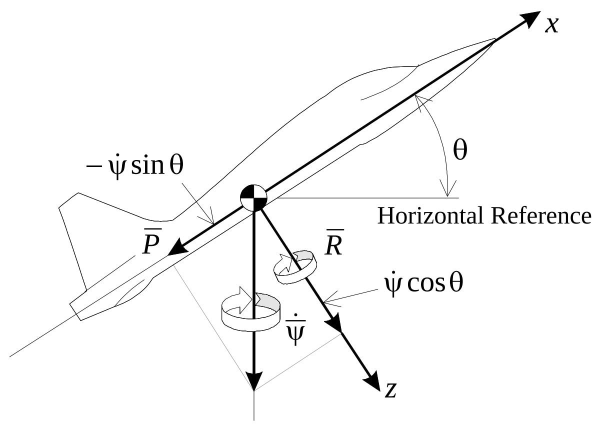


Figure 7.16 Development of Aircraft Angular Velocities by the Euler Angle Yaw Rate (θ rotation)

Finally, the body axis system is rotated about the *x*-axis with an angular velocity of $\dot{\phi}$ (Figure 7.17).

$$\begin{bmatrix} p_3 \\ q_3 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ \sin\theta & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} -\dot{\psi}\sin\theta + \dot{\phi} \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{bmatrix} = \begin{bmatrix} -\dot{\psi}\sin\theta + \dot{\phi} \\ \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi \end{bmatrix}$$

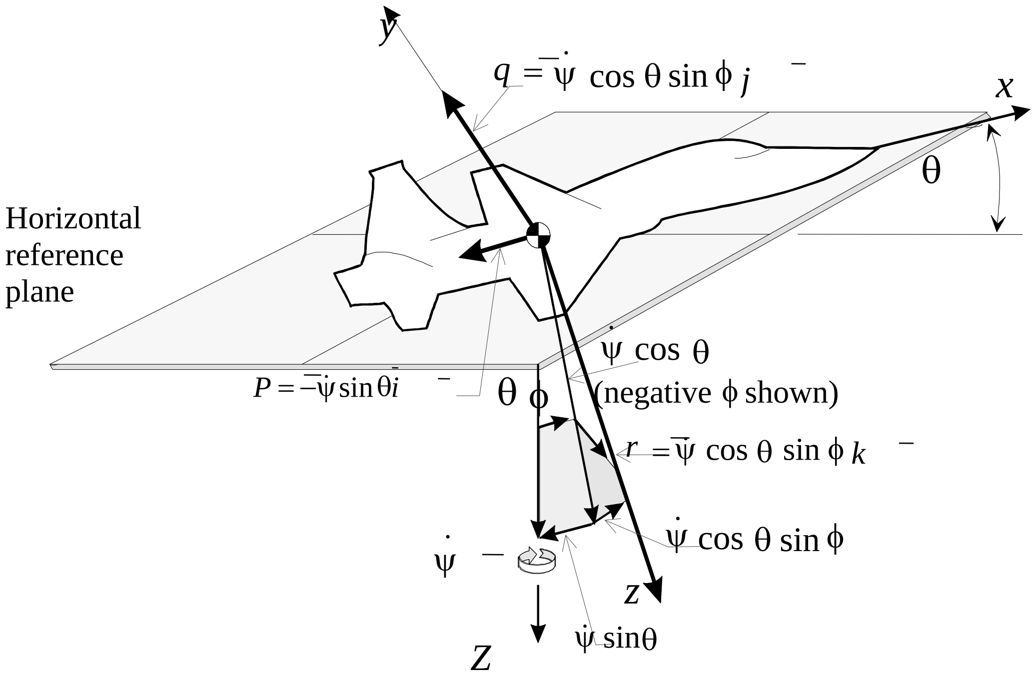


Figure 7.17 Development of Aircraft Angular Velocities by the Euler Angle Yaw Rate (ϕ rotation)

Therefore, the body-axis components of p, q, and r, in terms of inertial-axis rates and Euler angles are:

$$p = \dot{\phi} - \dot{\psi}\sin\theta \tag{7.12}$$

$$q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \tag{7.13}$$

$$r = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \tag{7.14}$$

Since $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ were about the z_0, y_1, and x_2 axes, they are not orthogonal. These equations can be solved explicitly for $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ since they are important forms of the equations. This can be accomplished by first writing the above equations in matrix form:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{7.15}$$

We can then invert the transformation matrix in the right-hand side of equation 7.15 using the method described in Vectors and Matrices (Chapter 4 & 5) and premultiplying both sides of the equation by the inverse to get an equation solved explicitly for $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$.

First let
$$[A] = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Then take the transpose of $[A]$

$$[A]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ -\sin\phi & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Replace each element of $[A]^T$ with its cofactor to get the adjoint matrix:

$$adj[A] = \begin{bmatrix} \cos\theta & \sin\theta\sin\phi & \sin\theta\cos\phi \\ 0 & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Finally, dividing through by the determinant of [*A*], gives the inverse matrix:

$$[A]^{-1} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}$$

Which gives the transformation matrix equation:

$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Therefore, the inertial-axis components of $\dot\phi$, $\dot\theta$, and $\dot\psi$ in terms of body-axis rates and Euler angles are:

$$\dot\phi = p + \tan\theta(q\sin\phi + r\cos\phi) \tag{7.16}$$

$$\dot\theta = q\cos\phi - r\sin\phi \tag{7.17}$$

$$\dot\psi = (q\sin\phi + r\cos\phi)/\cos\theta \tag{7.18}$$

The following list of angular rate and acceleration transformation is presented without proof. To avoid confusion, the subscript *b* denotes body axis, *s* denotes stability axis, and *w* denotes the relative wind axis system.

$$\dot p_b = \ddot\phi - \ddot\psi\sin\theta - \dot\psi\dot\theta\cos\theta$$

$$\dot q_b = \ddot\theta\cos\phi - \dot\theta\dot\phi\sin\phi + \ddot\psi\cos\theta\sin\phi - \dot\psi\dot\theta\sin\theta\sin\phi + \dot\psi\dot\phi\cos\theta\cos\phi$$

$$r_b = \ddot\psi\cos\theta\cos\phi - \dot\psi\dot\theta\sin\theta\cos\phi - \dot\psi\dot\phi\cos\theta\sin\phi - \ddot\theta\sin\phi - \dot\theta\dot\phi\cos\phi$$

$$\dot\alpha = q_b - (p_b\cos\alpha + r_b\sin\alpha)\tan\beta - \tfrac{g}{V_T\cos\beta}\left[\left(n_{z_b} - \cos\theta\cos\phi\right)\cos\alpha + \left(n_{x_b} - \sin\theta\right)\sin\alpha\right]$$

$$\dot\beta = p_b\sin\alpha - r_b\cos\alpha + \tfrac{g}{V_T}\left\{\left(n_{y_b} + \cos\theta\sin\phi\right)\cos\beta - \left[\left(n_{x_b} - \sin\theta\right)\cos\alpha - \left(n_{z_b} - \cos\theta\cos\phi\right)\sin\alpha\right]\sin\beta\right\}$$

$$p_s = p_b\cos\alpha + r_b\sin\alpha$$

$$\dot p_s = \dot p_b\cos\alpha + \dot\alpha p_b\sin\alpha + \dot r_b\sin\alpha + \dot\alpha r_b\cos\alpha$$

$$q_s = q_b$$

$$\dot q_s = \dot q_b$$

$$r_s = r_b\cos\alpha - p_b\sin\alpha$$

$$\dot{r}_s = \dot{r}_b \cos\alpha - \dot{\alpha} r_b \sin\alpha - \dot{p}_b \sin\alpha - \dot{\alpha} p_b \cos\alpha$$

$$p_w = p_s \cos\beta + q_s \sin\beta$$

$$\dot{p}_w = \dot{p}_s \cos\beta - p_s \dot{\beta} \sin\beta + \dot{q}_s \sin\beta + q_s \dot{\beta} \cos\beta$$

$$q_w = q_s \cos\beta - p_s \sin\beta$$

$$\dot{q}_w = \dot{q}_s \cos\beta - q_s \dot{\beta} \sin\beta - \dot{p}_s \sin\beta - p_s \dot{\beta} \cos\beta$$

$$r_w = r_s$$

$$\dot{r}_w = \dot{r}_s$$

7.4 Flight Path Angles

Just as the three Euler angles define the attitude of the aircraft with respect to the Earth, three flight path angles describe the vehicle's *cg* trajectory relative to the Earth (not the air mass).

- σ = Flight path heading angle; also known as ground track heading, is the horizontal angle between some reference direction (usually North) and the projection of the velocity vector on the horizontal plane. Positive rotation is from North to East.
- γ = Flight path elevation angle; the vertical angle between the velocity vector and the horizontal plane. Positive rotation is up. During a descent, this parameter is commonly known as glide path angle.
- μ = Flight path bank angle; the angle between the plane formed by the velocity vector and the lift vector and the vertical plane containing the velocity vector. Positive rotation is clockwise about the velocity vector, looking forward.

The first two parameters above are easily measured using ground-based radar or onboard GPS or inertial reference systems. If only α, β, and the Euler angles are available, then **assuming zero winds**, the flight path angles can be calculated as

$$\gamma = \sin^{-1}\left[(\sin\theta\cos\alpha - \cos\theta\cos\phi\sin\alpha)\cos\beta - \cos\theta\sin\phi\sin\beta\right]$$

$$\sigma = \sin^{-1}\left(\frac{\left[-\sin\phi\sin\alpha\cos\beta + \cos\phi\sin\beta\right]\cos\psi + \left[(\cos\theta\cos\alpha + \sin\theta\cos\phi\sin\alpha)\cos\beta + \sin\theta\sin\phi\sin\beta\right]\sin\psi}{\cos\gamma}\right)$$

$$\mu = \sin^{-1}\left(\frac{\cos\theta\sin\phi\cos\alpha + \sin\theta\sin\alpha}{\cos\gamma}\right)$$

The rate for each of these parameters can be calculated as follows:

$$\dot{\sigma} = \tfrac{g}{V_T\cos\gamma}\left[-n_x(\cos\alpha\sin\beta\cos\mu - \sin\alpha\sin\mu) + n_y(\cos\beta\sin\mu) + n_z(\sin\alpha\sin\beta\cos\mu + \cos\alpha\sin\mu)\right]$$

$$\dot{\gamma} = \tfrac{g}{V_T}\left[n_x(\cos\alpha\sin\beta\sin\mu + \sin\alpha\cos\mu) - n_y(\cos\beta\sin\mu) - n_z(\sin\alpha\sin\beta\sin\mu - \cos\alpha\cos\mu) - \cos\gamma\right]$$

$$\dot{\mu} = (p\cos\alpha + r\sin\alpha)\sec\beta + \dot{\sigma}\sin\gamma + \tfrac{g}{V_T}(n_x\sin\alpha + n_z\cos\alpha - \cos\mu\cos\gamma)\tan\beta$$

Actually, these equations describe the **velocity vector** (angles relative to the air mass). If the air mass is moving relative to the Earth, as is usually the case, the above equations <u>do not</u> describe the flight path.

7.5 Transforming Motions Through Axes

It is quite common to measure vehicle motions such as velocities, accelerations and rates through body-mounted instrumentation. It is also often useful to know how these motions are converted to another coordinate system. Figure 7.18 shows reference system *xyz* with some velocity relative to the *XYZ* axes. ω is the total angular velocity of the *xyz* coordinate system.

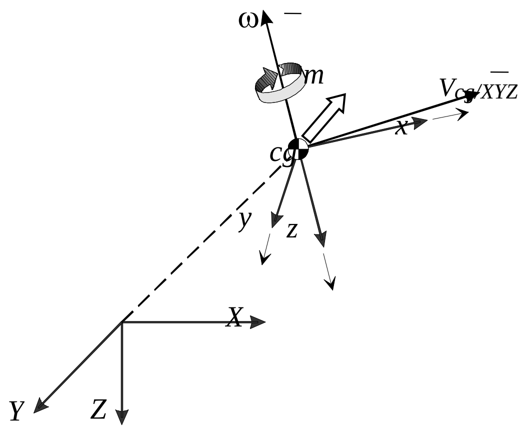


Figure 7 .18 True Velocity of Body with Respect to Fixed-earth Axes Coordinate Systems

7.5.1 Velocity Transformation

Considering some rigid body with an element *m*, vector analysis shows that the velocity of *m* as seen from an outside reference (*X, Y, Z*), is the sum of the body's *cg* velocity relative to that same reference plus the body's angular velocity acting through the distance between *m* and the *cg*:

$$\overline{V_T}_{m/XYZ} = \overline{V_T}_{XYZ} + \left[\overline{\omega} \times \overline{r}\right]$$

$\overline{V_T}$, $\overline{\omega}$, and $\overline{r}$ are vectors defined as:

$$\overline{V_T}_{XYZ} \equiv u\hat{i} + v\hat{j} + w\hat{k} \tag{7.19}$$

$$\overline{\omega} \equiv p\hat{i} + q\hat{j} + r\hat{k} \tag{7.20}$$

and:

$$\overline{r} \equiv x\hat{i} + y\hat{j} + z\hat{k} \tag{7.21}$$

Combining the four equations above into matrix form gives:

$$\overline{V}_{TXYZ} = u\hat{i} + v\hat{j} + w\hat{k} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ x & y & z \end{vmatrix}$$

Expanding yields:

$$\overline{V}_{Tm/XYZ} = \left[u\hat{i} + v\hat{j} + w\hat{k} + (qz - ry)\hat{i} - (pz - rx)\hat{j} + (py - qx)\hat{k}\right]$$

Rearranging gives:

$$\overline{V}_{Tm/XYZ} = (u + qz - ry)\hat{i} - (v + rx - pz)\hat{j} + (w + py - qx)\hat{k} \tag{7.22}$$

The *XYZ* in $\overline{V}_{TXYZ}$ is any other reference system, typically the inertial coordinate system. In this case, it can be considered as the total velocity of the element. Equation 7.22 shows that the total velocity of any point on a moving body can be described by arranging component velocities according to the body's axes (a.k.a. unit vectors). Not only can there be linear velocity along each body axis, but an additional "coupled velocity" can exist. This comes from an angular rate acting at some distance from the *cg*.

An example of a coupled velocity is seen by an earth-based (inertial) observer who examines the right wingtip of an aircraft yawing to the right. The positive yaw rate and positive lateral position combine to give a tip motion towards the rear of the aircraft, a negative velocity. This is shown as the "$-ry$" term in Equation 7.22.

7.5.2 Acceleration Transformation

From vector analysis, the derivative of the velocity $\overline{V}_T$ in the inertial (fixed-earth) coordinate system is related to the derivative of $\overline{V}_T$ along the body axis system through the relationship:

$$\left[\frac{d\overline{V}_T}{dt}\right]_{XYZ} = \left[\frac{d\overline{V}_T}{dt}\right]_{xyz} + \overline{\omega} \times \overline{V}_T \tag{7.23}$$

Using Equations 7.19 and 7.20, the acceleration equation (7.23) can be written as:

$$\left[\frac{d\overline{V}_T}{dt}\right]_{XYZ} = \dot{u}\hat{i} + \dot{v}\hat{j} + \dot{w}\hat{k} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ u & v & w \end{vmatrix}$$

Expanding and rearranging terms gives:

$$\left[\frac{d\overline{V}_T}{dt}\right]_{XYZ} = \left[\left(\dot{u} + qw - rv\right)\hat{i} + (\dot{v} + ru - pw)\,\hat{j} + (\dot{w} + pv - qu)\hat{k}\right]$$

This total transational acceleration can also be broken down into three components according to the body's unit vectors

$$a_x = \dot{u} + qw - rv \tag{7.24}$$

$$a_y = \dot{v} + ru - pw \tag{7.25}$$

$$a_z = \dot{w} + pv - qu \tag{7.26}$$

These equations show how motions observed from a moving platform relate to accelerations in inertial space. Consider Equation 7.22 in the case of an aircraft performing a level acceleration maneuver: both pitch rate and yaw rate are zero, so the equation simplifies to $a_x = \dot{u}$. This seems straightforward enough, but if the aircraft is also yawing and translating sideways, then *r* and *v* combine to create an additional acceleration not accounted for by the simple $\dot{u}$ term. This is another cross-coupled effect. This effect can be visualized by imagining a top view of an aircraft with its *cg* moving right (positive *v*) while yawing right (positive *r*). The ensuing acceleration of the *cg* due to these motions is opposite to the overall aircraft acceleration $(\dot{u})$. Equation 7.24 also shows this to be a negative acceleration along the aircraft's *i* component. A similar effect occurs with each of the cross-coupled terms in the equations above.

Acceleration of the aircraft's flight path is used to describe turn capability and other kinematic information. Flight path acceleration is identical to acceleration (or load factor) along each axis of the relative wind axis system. The following transformation correlates linear acceleration from the (measurable) body-axis system to the wind axis system. Note that they are just a specific application of Equations 7.1 and 7.2.

$$\begin{bmatrix} N_{xw} \\ N_{yw} \\ N_{xw} \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} N_{xb} \\ N_{yb} \\ N_{xb} \end{bmatrix} \tag{7.27}$$

The inverse of this matrix is easily shown to be

$$\begin{bmatrix} N_{xb} \\ N_{yb} \\ N_{xb} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0-\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_{xw} \\ N_{yw} \\ N_{xw} \end{bmatrix}$$

The chapter on inertial navigation systems combines the transformations from Equations 7.10 and 7.27 to convert initially sensed motions & angles to flight path motions.

7.5.3 Specific Angular Momentum

When working with the linear motions above, acceleration in any direction can be considered a "specific force" or *F/m*. If an object's specific force is multiplied by its mass, a force would result along the same direction [$(F/m) \times m = Force$]. There is an equivalent to this simple idea when working with moments. Basic kinematics states that angular momentum (*H*) is linear momentum (*mV*) acting through a moment arm (*r*) or:

$$\overline{H} = \overline{r} \times m\overline{V} = \left(\overline{r} \times \overline{V}\right)m \tag{7.29}$$

Simply dividing this equation through by *m* yields the specific angular momentum, $\left[\frac{H}{m}\right] = \overline{r} \times \overline{V}$. Just like the case with specific force, multiplying the specific angular momentum of any point by a mass gives the total angular momentum of that mass. The cross product $\left(\overline{r} \times \overline{V}\right)$ can be thought of as a ball swinging on the end of a string as shown in Figure 7.19. If the angle between $\overline{r}$ and $\overline{V}$ is 90 degrees, as shown in the figure, then the product is simply $r\overline{V}$.

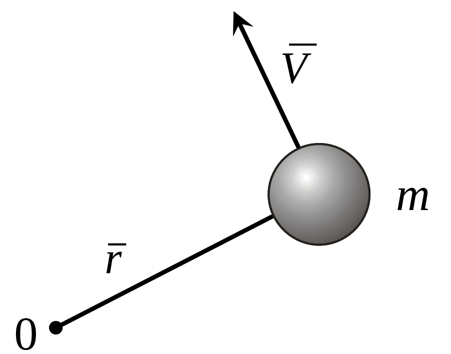


Figure 7 .19 Specific Angular Momentum

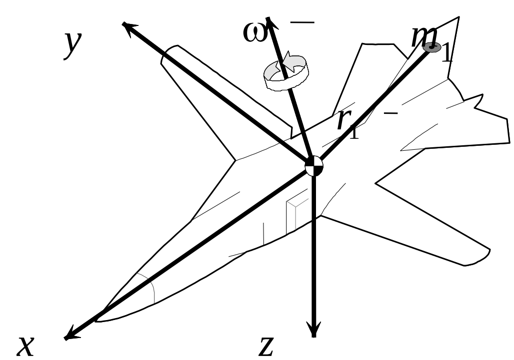


Figure 7 .20 Position vector of Vehicle element

To determine the specific angular momentum of any spot on a moving coordinate system, consider a small element (m_1) located some distance from an aircraft's *cg* - represented by the position vector r_1 as shown in Figure 7.20.

The specific angular momentum of m_1 is: $$\left[\frac{H}{m}\right] = \overline{r}_1 \times \overline{V}_1 \tag{7.30}$$

First consider the above velocity term. From vector analysis, the rate of change of the radius vector r_1 (V_1) can be related to the body axis system by:

$$\overline{V}_1 = \left[\frac{d\,\overline{r1}}{dt}\right]_{XYZ} = \left[\frac{d\,\overline{r1}}{dt}\right]_{xyz} + \overline{\omega} \times \overline{r}_1$$

Assuming the aircraft is a rigid body, then r_1 does not change with time and the first term can be excluded. The above equation then simplifies to:

$$\overline{V}_1 = \overline{\varpi} \times \overline{r}_1$$

Applying matrix algebra and Equations 7.20 and 7.21 again gives

$$\overline{\varpi} \times \overline{r}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ x & y & z \end{vmatrix}$$

Which can be expanded to

$$\overline{\omega} \times \overline{r}_1 = (qz - ry)\hat{i} + (rx - pz)\hat{j} + (py - qx)\hat{k}$$

This can be inserted into Equation 7.30 to give the specific angular momentum for any element m_1:

$$\left[\frac{H}{m}\right] = \overline{r}_1 \times \overline{V}_1 = \overline{r}_1 \times [\overline{\varpi} \times \overline{r}_1] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ z & y & z \\ (qz - ry) & (rx - pz) & (py - qx) \end{vmatrix}$$

Finally, this matrix can be expanded to show the specific angular momentum of any single element for each of the three body axis components.

$$\left[\frac{H}{m}\right]\hat{i} = y(py - qx) - z(rx - pz) = p(y^2 + z^2) - q(xy) - r(xz) \quad (7.31)$$

$$\left[\frac{H}{m}\right]\hat{j} = z(pz - qy) - x(py - qx) = q(x^2 + z^2) - r(yz) - p(xy) \quad (7.32)$$

$$\left[\frac{H}{m}\right]\hat{k} = x(rx - pz) - y(qz - ry) = r(x^2 + y^2) - p(xz) - q(yz) \quad (7.33)$$

The units for each term above is ft^2/sec or [*ft/sec*]× *ft*, i.e.; velocity about a moment arm as shown in Figure 7.19. The total angular momentum for a real object can be calculated by adding up the angular momentum for each mass element. Another way to determine this is to integrate Equations 7.31 - 7.33 across the density of the object. This will be presented when developing the aircraft Equations of Motion, Chapter 4.

7.6 Summary

This chapter defines the most common axis systems used in aeronautics and establishes the practices used for transforming motion from one system to another. These procedures can be extended to other axis systems such as "North, East, Down" as done in Chapter 2, Inertial Navigation Systems. Many transformations exist which can be used to "simulate" parameters that cannot be practically instrumented directly. The scope of this text is limited to the most common and useful transformations required for developing the equations of motion and other basic flight information.

7.7 References

Anon., *Aircraft Flying Qualities, Chapter 4, Equations of Motion*, USAF Test Pilot School notes, AFFTC Edwards AFB CA, March 1991.

Kalviste, Juri, *Flight Dynamics Reference Handbook*, Northrop Corp. Aircraft Division, April 1988.

Volume 1 – Math & Physics for Flight Testers

Chapter 8

Mechanics

Table of Contents

8.1 Preface

A knowledge of mechanics is essential to understanding' the modern aircraft and aircraft systems. Test pilots and flight test engineers must have a thorough understanding of the physical concepts of forces and the effects of forces on bodies to be able to apply mathematical modeling techniques so that an insight can be obtained on the effects of the large number of variables on the flight test results. The science of engineering is founded on physical laws expressed in the form of mathematical equations. Both subjects are so closely related that one never encounters one without the other; however, with the recent advances in digital computations and the use of personal computers, the primary attention appears to be focused on the mathematical framework of mechanics with much less attention to physical reality and engineering applications. The result of this recent trend is that there are many flight testers whose engineering knowledge is restricted to that which is stored in a computer and whose analytical reasoning is suppressed by computer usage.

The mechanics of aircraft and weapons systems is a narrow application of the basic laws of physics and, if these physical laws and concepts are not well understood at the beginning of the test pilot school curriculum, the student will have a difficult time understanding the dynamics of flight vehicles, the flight test techniques and the data analysis.

This review of mechanics includes only material which is of use in the test pilot curriculum. Most of the subjects can be found in standard university text books. The advantages of this publication over other texts is the elimination of material that does not directly apply to flight vehicles and most of the examples are aircraft related. These notes are the result of teaching U.S. Army pre-TPS pilot students in the basics of engineering concepts at Edwards AFB prior to going to Test Pilot School.

8.2 Introduction

Mechanics deals with the relationship between forces, matter and motion. In these notes, all bodies will be considered as being rigid and only the effects of forces on the motion of the body will be considered. The effects of forces on the shape of the bodies is a special area of study called strength of materials which, although very important in the design of aircraft structures, is not considered in these notes.

A knowledge of mechanics provides the tools for an understanding of the factors involved in the motion of ground vehicles, aircraft and aircraft weapons systems and is therefore extremely important for the test pilot and flight test engineer student.

8.2.1 Units of Measurement

To adequately define a physical phenomenon such as a velocity for example, it is necessary that a magnitude and a dimension be used. A velocity of 100 is meaningless unless accompanied by ft/sec, or miles per hour, etc... All quantities in mechanics have dimensions which can be expressed as combinations of three basic units:

distance: (L), mass: (M). time: (T)

For example, velocity, which is defined as the change in distance divided by the change in time, has the dimension expressed by the general units of distance divided by time, or L/T as shown by the dimensions of feet per second, miles per hour or knots. Since there are many arbitrary methods of measuring the same physical quantities, one system must be selected that best fits the needs of the flight test community in the United States. The English Gravitational System has been selected which uses feet for distance, seconds for time and slugs for mass. The gravitational system in which length, force and time are considered fundamental quantities and the units of mass are derived will give the same results as the absolute system in which length, mass and time are considered fundamental and the units of force are derived. Engineers prefer to use force as a fundamental quantity because most experiments involve the direct measurement of force. The slug is the unit of mass in the British engineering's system and is the mass of a body which weighs 32.2 lb. at the earth surface. The pound force is the force required to accelerate a slug mass at a rate of 1 ft/sec^2. The pound force may also be defined as the gravitational attraction exerted by the earth on a one pound mass at sea level. The acceleration due to the earth's attraction (gravity) is taken as 32.2 ft/sec^2 at approximately the 45° latitude position on the earth, hence the relationship/between the pound mass and the slug.

Despite the rather lengthy discussion on axis systems used in aircraft work, most of the work in this text is based on the fixed earth axis system and a large majority of the problems are only one-dimensional, i.e., straight line motion.

Time	is a measure of the succession of events and is considered an absolute quantity. The unit of time is the second which is a convenient fraction of the 24 hour day.
Force	is defined as a push or pull on a body and is measured in pounds (lb) and has the dimensions of $M\ L/T^2$. A force tends to move a body in the direction of the applied force.
Pressure	is the effect of a force on a unit area of a body and is expressed in pounds (lb) per square foot (lb/ft) $Presure = \frac{Force}{Area} = \frac{F}{A}$ and pressure has the dimensions of M/T^2L
Matter	is substance which occupies space. A *body* is matter bounded by a closed surface.
Inertia	is a body property and it is the resistance of the body to changes in motion.
Mass	is the quantitative measure of inertia of all bodies and has the units of a slug.
Particle	is a body of negligible dimensions and can be considered as a point mass.
Rigid Body	A body is said to be rigid when no deformation of the body occurs when forces or moments are applied. All bodies deform when under loads and is the subject of strength of materials', however, in most aircraft flight test problems, with the exception of aeroelastic phenomena the deformation of the aircraft is very small relative to the aircraft motion and the aircraft is considered a rigid body.
Scalar	is a quantity which has a magnitude only. Examples are time, volume, mass, density...
Vector	is a quantity which has a direction as well as a magnitude. Examples are velocity, displacement, acceleration, force, moment and momentum.
Accuracy	The number of significant figures shown in a numerical calculation should he no greater than the number of figures which can be justified by the accuracy of the given data. Hence, the cross-sectional area of a square bar whose side is 0.24 ins. measured to the nearest hundredth of an inch should be written as 0.058 in^2 and not 0.0576 in^2. When calculations involve small differences between large quantities, greater accuracy must be achieved. For example, it is necessary to know the altitudes 42, 503 ft. and 42, 391 ft. to an accuracy of five significant digits in order that the difference of 112 ft. can be expressed to three figure accuracy.

8.3 STATICS

8.3.1 Introduction

Statics is a study of forces and moments without motion. The forces and moments can either be in equilibrium, i.e. $\Sigma F = \Sigma M = 0$ or there can be a resultant force and moment acting on the body.

A force is a vector quantity in that it has a magnitude in pounds (lb) force and a direction. A moment is a force times a distance i.e. Moment = $F.l$

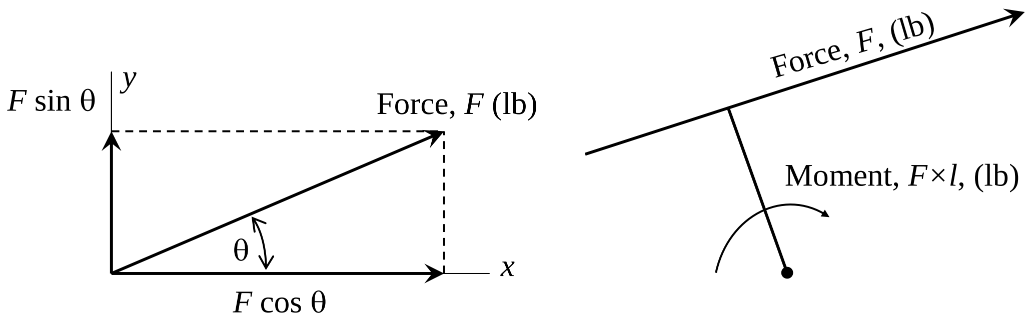


Figure 8.1. Definition of Forces and Moments

Forces can be resolved into two components at ninety degrees to each other. In Figure 8.1., the vertical component of the force (F) at an angle (θ) to the x axis is $F \sin \theta$ and the horizontal component is $F \cos \theta$. Alternatively, the sum of the two force vectors. $F \sin \theta$ and $F \cos \theta$ equals the force (F), which demonstrates how force vectors are added.

8.3.2 Concurrent Forces

Concurrent forces are forces that act at a single point as shown in Figure 8.2. The resultant force could be determined by adding vectorially F_1 and F_2. then adding vectorially the resultant from the sum of F_1 and F_2 to the third force, F_3. This approach is awkward and increases in difficulty with an increase in the number of forces. A simple approach is to resolve each force into a vertical and horizontal component, then add all the vertical and horizontal components algebraically, then determine the vector sum of the resultant vertical and horizontal components.

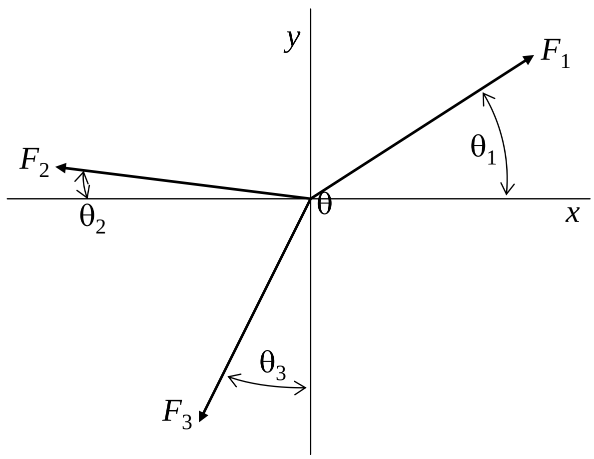


Figure 8.2 Concurrent Forces

For example, find the resultant force in Figure 8.2. Resolve the forces vertically:

$$\uparrow F_1 \sin \theta_1 + F_2 \sin \theta_2 - F_3 \sin \theta_3 = F_{RV}$$

where the arrow ↑ indicates that up is positive and F_{RV} is the resultant vertical force.

Resolve horizontally →

$F_1 \cos\theta_1 - F_2 \cos\theta_2 - F_3 \sin\theta_3 = F_{RH}$

where the arrow → indicates that the direction to the right is positive. The vector sum of F_{RV} and F_{RH} is then determined as shown to find F_R, the resultant force in the direction θ_R from the horizontal axis, where;

$$F_R = \sqrt{(F_{RV})^2 + (F_{RH})^2} \text{ and } \tan\theta_R = \frac{F_{RV}}{F_{RH}}$$

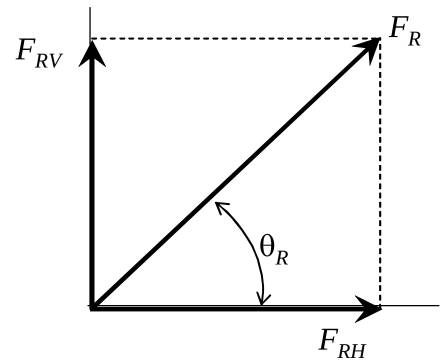


Figure 8.3 Vector Sum of Two Forces

8.3.3 Parallel Forces

When the forces acting on a body are parallel, then not only must the resultant force be parallel to the force components, but location of the resultant force on the body is unique, i.e., has a single value.

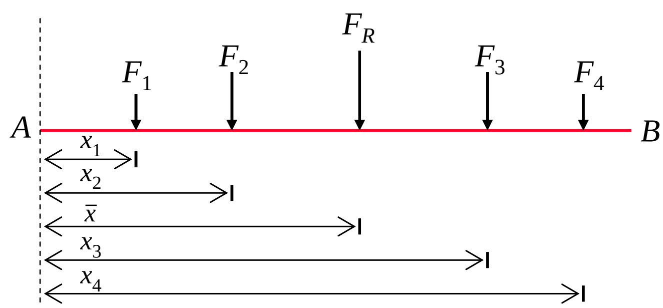


Figure 8.4. Parallel Forces Acting on a Beam.

From Figure 8.4, it is obvious that the resultant force F_R has the value:

$$F_R: F_1 + F_2 + F_3 + F_4$$

However, to determine the location, of the resultant force F_R, we must take moments about a convenient point on the beam. For the purpose of illustration, take moments about $\overrightarrow{A}$, the arrow indicating that moments to the right are positive.

$$\overrightarrow{A} = F_1 x_1 + F_2 x_2 + F_3 x_3 + F_4 x_4$$

Since the summation of the moments of all of the forces must equal the moment generated by the resultant force, the location of the resultant force is defined as $\bar{x}$, i.e.,

$$\overrightarrow{A} = F_R \bar{x} = (F_1 + F_2 + F_3 + F_4)\bar{x}$$

therefore,

$$F_1 x_1 + F_2 x_2 + F_3 x_3 + F_4 x_4 = (F_1 + F_2 + F_3 + F_4)\bar{x}$$

and

$$\bar{x} = \frac{F_1 x_1 + F_2 x_2 + F_3 x_3 + F_4 x_4}{F_1 + F_2 + F_3 + F_4}$$

or in more general terms, the distance

$$\bar{x} = \frac{\sum Moments}{\sum Forces}$$

Note: The same result can be obtained by taking moments about any point on the beam, eg, *B*

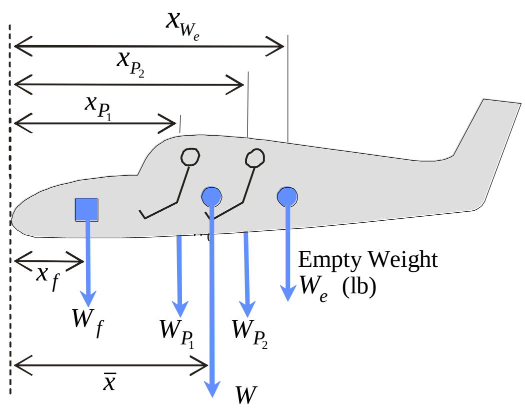


Figure 8.5 Determination of Aircraft Center of Gravity.

The determination of the resultant of parallel forces is directly applicable to the determination of the center of gravity of aircraft. On an aircraft, the moments are determined about the aircraft's reference datum line and all expendables, crew and cargo are located with reference to the datum line as shown in Figure 8.5.

Taking moments about the reference datum, then,

$$\bar{x} = \frac{\sum Moments}{\sum Weights} = \frac{(W_e)(x_{W_e}) + (W_{P_1})(x_{P_1}) + (W_{P_2})(x_{P_2}) + (W_f)(x_f)}{W_e + W_{P_1} + W_{P_2} + W_f}$$

8.3.4 Non Concurrent and Non Parallel Forces

If a system of forces is acting on a body, for example a circular disc as shown in Figure 8.6, then to determine the resultant force and its position on the disc, the following procedure can be, used;

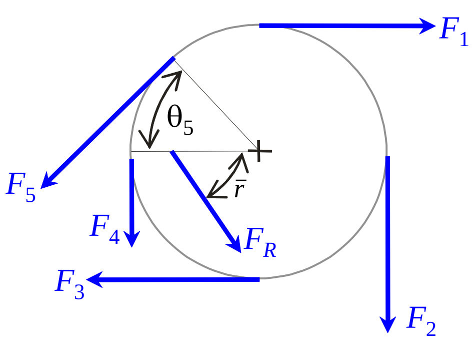


Figure 8.6 Forces on a Disc

a. Resolve all the forces into horizontal and vertical components.

b. Determine the algebraic- sum of the vertical and horizontal components.

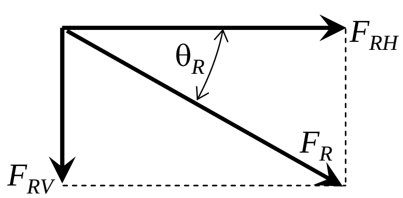
c. From the sum of the two components sums, find the magnitude and direction of the resultant force F_R.

d. Take moments about any convenient location to find the distance of the F_R from the center.

a. Forces → : $F_1 - F_3 - F_5 \sin\theta_5 = F_{RH}$. →

b. Forces ↑ : $- F_2 - F_4 - F_5 \cos\theta_5 = F_{RV}$. ↑

c.



$$F_R = \sqrt{(F_{RV})^2 + (F_{RH})^2} \text{ and}$$

$$\tan\theta_R = \frac{F_{RV}}{F_{RH}}$$

d. $\vec{c}: +(F_1 r) + (F_2 r) + (F_3 r) - (F_4 r) - (F_5 r) = (F_R \bar{r})$

therefore $\bar{r} = \dfrac{(F_1 + F_2 + F_3 - F_4 - F_5) r}{F_R}$

8.3.5 Transfer of a Force to a New Location.

If a. force is acting on a body, for example, the *resultant aerodynamic force* (*RAF*) that acts on an airfoil at the centre of pressure, as shown in **Figure 8.7**, then this *RAF* can be moved to another arbitrary position if the following conditions apply.

a. The new force is parallel and equal to the original force.

b. A moment (*m*) is applied, about the new location such that the sum of the moments about any point on the airfoil of the original RAF equals the new transferred system of force and moment.

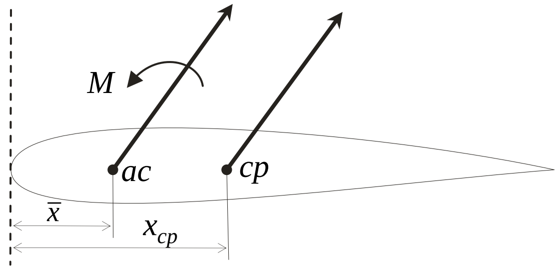


Figure 8.7 Forces on an Airfoil

For airfoils it is customary to transfer the resultant aerodynamic force from the centre of pressure to the aerodynamic centre which is located at the 25% point subsonically. The *aerodynamic center* (*ac*) of an airfoil is located on an airfoil such that the moment (*M*) is constant and is not dependent upon angle of attack.

8.3.6 Mass, Weight, Centre of Gravity and Moment of Inertia.

The mass of a body is its volume times its density, i.e. (volume) × (density). The weight of a body is the force acting on the mass of the body due to the gravitation effect of the earth. The weight vector always acts towards the centre of the earth and is equal to the mass of the body times the acceleration due to gravity. i.e.

$$Weight = (Mass)\,(Acceleration\ due\ to\ Gravity)$$
$$W = mg$$

The location of the weight vector in the body is called the center of gravity. Alternatively the centre of gravity (*cg*) is defined as the point through which the weight vector acts to accelerate the mass towards the center of the earth without the body rotating.

Newton's second law states that

$$Force = (Mass)\,(Acceleration)$$
$$F = mA$$
$$F = \frac{W}{g}a$$

Therefore if we know the out of balance force (F) acting on a body of weight (W), we can calculate the acceleration of that body due to the force (F).

In rotary motion where a body is rotating about a centre of rotation then Newton's second law is modified by multiplying both sides of the equation by a length *l* which is the distance from the force to the centre of rotation, i.e. a moment.

$$F \times l = Moment = M \times l \times a$$

however a linear acceleration *a* that applies to a mass rotating about a center of rotation, then,

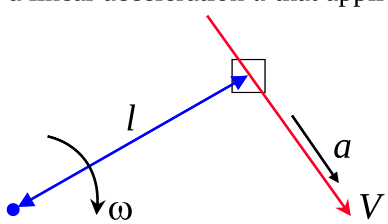


Figure 8.7 Mass rotating about a Center of

$a = \alpha\, l$ for acceleration

$V = \omega\, l$ for velocity

where ω = angular velocity in radians/sec

and α = angular acceleration in rads/sec

Rotation.

Therefore, Newton's second law is now :

$$\text{Moment} = (m\,l^2)\,\alpha$$

and ($m\ l^2$) is defined as the mass moment of inertia or alternatively the moment of inertia of a mass rotating about an axis is the mass times the square of the distance from the centre of rotation.

$$\text{Moment} = I\,\alpha$$

Where I is the mass moment of Inertia.

8.3.7 Centroids, Centre of Gravity & Moments of Inertia

A study of flat plates of uniform thickness in which the weight is proportional to the plate area is interesting and demonstrates the principles discussed earlier.

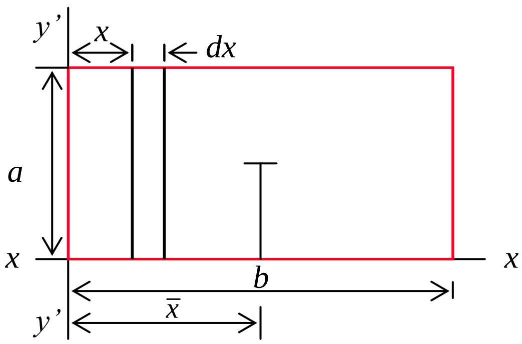


Figure 8.8

The centroid of a uniform, homogeneous flat plate is the equivalent to the centre of gravity of a body.

To determine the horizontal location of the centroid of a flat plate of dimensions (a) (b) as shown in Figure 8.8, take moments about $y'\,y'$ of the strip a dx

$$\int_0^b (a\,dx)x = \bar{x}\int_0^b a\,dx$$

or

$$\bar{x} = \frac{\int_0^b ax\,dx}{\int_0^b a\,dx} = \frac{a\left[\frac{x^2}{2}\right]_0^b}{a[x]_0^b} = \frac{b}{2}$$

Similarly, $\bar{y}$ can be found to be $\frac{a}{2}$ from the axis xx. The moment of inertia of the flat plate shown in Figure 8.8 about the yy axis is ;

$$I_{yy} = \int_0^b (a\,dx)x^2 = a\left[\frac{x^3}{3}\right]_0^b = \frac{ab^3}{3}$$

Similarly,

$$I_{xx} = \int_0^a (b\,dy)y^2 = b\left[\frac{y^3}{3}\right]_0^a = \frac{ba^3}{3}$$

Example: Find the moments of inertia of a flat plate shown in Figure 8.9 through the centroid, $\frac{b}{2}$, $\frac{a}{2}$.

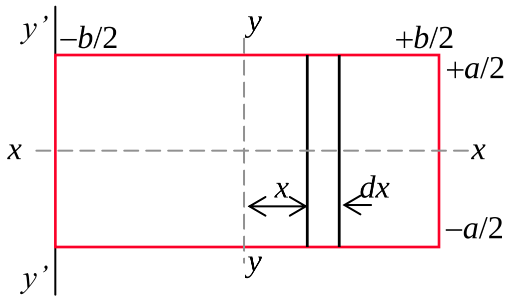


Figure 8.9 Flat Plate

$$I_{yy} = a\int_{-b/2}^{+b/2} x^2 dx = a\left[\frac{x^3}{3}\right]_{-b/2}^{+b/2} = \frac{ab^3}{12}$$

$$I_{xx} = \int_{-a/2}^{+a/2} y^2 dy = b\left[\frac{y^3}{3}\right]_{-a/2}^{+a/2} = \frac{ba^3}{12}$$

8.3.8 Parallel Axis Theorem

The parallel axis theorem states that the moment of inertia about any axis equals the moment of inertia about the center of gravity (centroid) plus the product of the mass (area) and. the square of the distance between the parallel axes.

Example: Find the $I_{y'y'}$ of the flat plate shown in Figure 8.9, using the parallel axis theorem.

$$I_{y'y'} = I_{yy} + (a \times b)\left(\frac{b}{2}\right)^2$$

$$= ab^3 + (a \times b)\left(\frac{b}{2}\right)^2 = \frac{ab^3}{3}$$

which agrees with the results obtained earlier.

8.3.9 Radius of Gyration

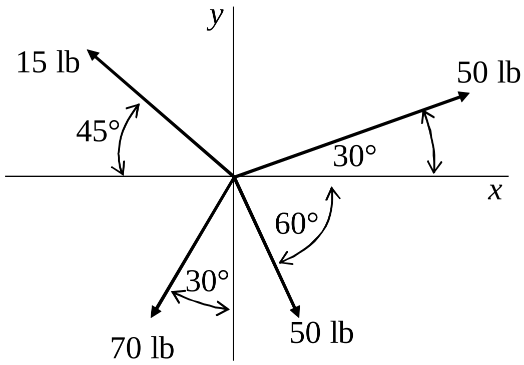
The radius of gyration is a number which when squared and multiplied by mass (area), will equal the moment of inertia about a specific axis.

$$I_{aa} = K_{aa}^2 A$$

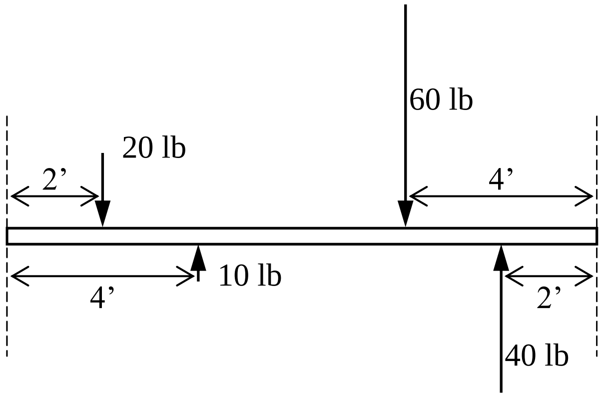
where K is the radius of gyration.

8.3.10 Tutorials

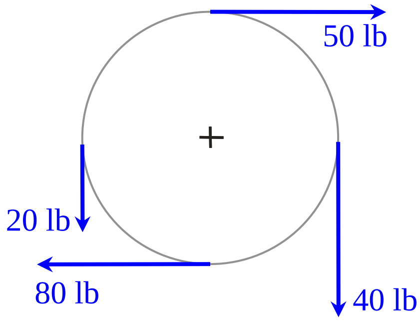
1. Find the resultant of the forces shown.



2. Find the resultant of the forces shown on the uniform 10 ft. beam. The beam weighs 100 lb.

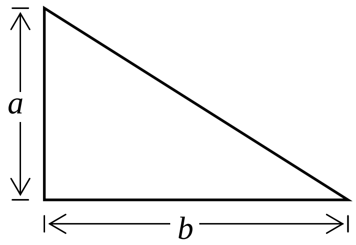


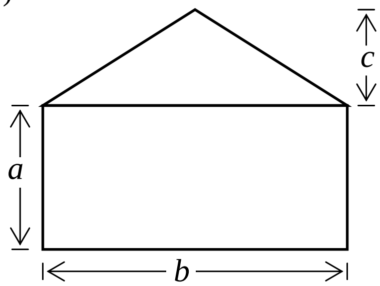
3. Find the resultant of the forces acting on a circular plate of radius 3 ft.

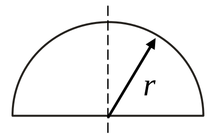


4. Find the centroids of the following shapes

a) b) c)

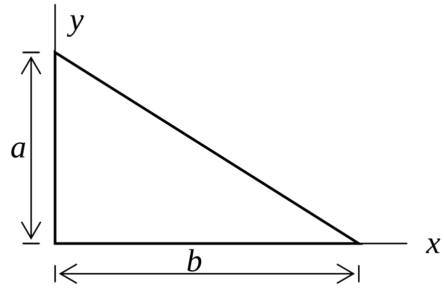




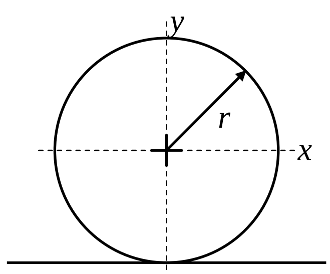


5. Find the moments of inertia of the following plates:

a) Find I_{yy} and I_{xx}

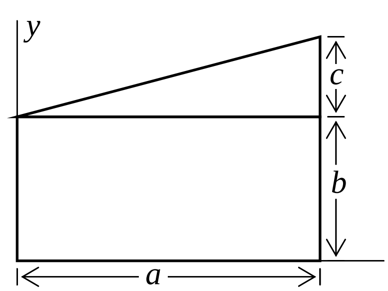


b) Find I_{xx} and $I_{x'x'}$



c) Find I_{zz} of the circle in b) above, where the axis 'zz' is perpendicular to the plane $x\ y$.

d) Find I_{xx} and I_{yy}



8.4 Friction

8.4.1 Introduction

Friction is essentially a loss of energy due to the relative movement between two bodies. The most usual ones encountered are shown in Figure 2. 1

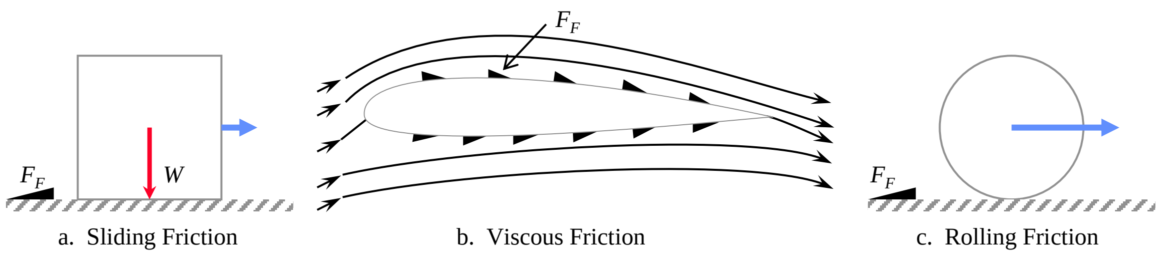


Figure 2.1 Examples of Friction Forces

8.4.2 Sliding Friction

Frictional forces always oppose the motion and sliding friction is the force that opposes the motion of one block sliding on a surface, as shown in Figure 2.1a. The sliding frictional force is a function of the normal force between the surfaces and the coefficient of friction between the block and the surface. Interestingly the frictional force is not a function of the contact area between the block and the surface.

Therefore, $$F_F = \mu R_N \tag{2.1}$$

where R_N is the normal reaction between the surfaces and μ is the coefficient of friction between the surfaces and is a function of smoothness, lubrication, etc...

8.4.3 Viscous Friction

Viscous forces are exerted between a body such as an airfoil and a fluid (liquid or gas) in which the body is immersed and is moving- relative to the fluid as shown in Figure 2.1b. The viscous forces of a fluid moving over a body are confined to a very small region in close proximity to the body called the boundary layer. The boundary layer is the region where the velocity of the fluid changes with respect to the distance from the surface. Figure 2.2 shows the velocity gradient of a laminar and turbulent boundary layer.

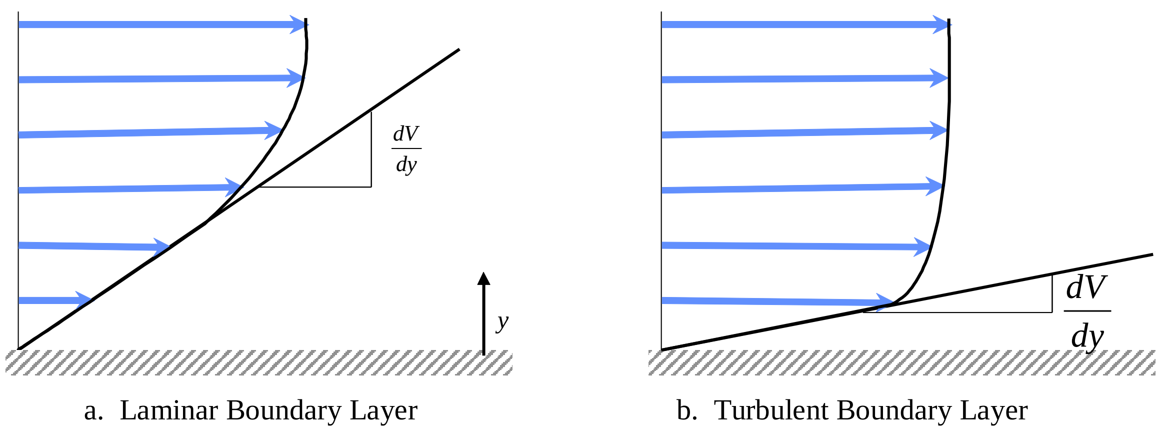


Figure 2.2 Laminar and Turbulent Boundary Layer Velocity Profile

A laminar boundary layer is a shear layer in which the individual shear layers are moving relative to each other without any vertical miring. The potential flow velocity which is the fluid velocity at the outer edge of the boundary layer is progressively slowed down and brought to rest at the surface. The velocity gradient *dV*/*dy* in a laminar boundary layer is low; therefore, the viscous shearing stress is low. The shearing stress, in Ib/ft^2 , is a function of the coefficient of viscosity μ and the velocity gradient *dV*/*dy*.

$$\tau = \mu \frac{dV}{dy} \frac{lb}{ft^2} \tag{2.2}$$

A turbulent boundary layer is a shear layer in which there is a vertical intermixing of the shear layers with the result that the velocity gradient at the surface is considerably greater than the velocity gradient at the surface of a laminar B.L. A turbulent B.L. has a considerably higher skin friction viscous drag than a laminar B. L. However, a turbulent B.L. also has more energy than a laminar B.L. and can therefore stay attached longer to an airfoil in an adverse pressure gradient.

8.4.4 Rolling Friction

The coefficient of friction of aircraft tires on a runway is a function of runway surface condition, tire material composition, tread, inflation pressure, surface friction shearing stress, relative slip speed, etc-.. When the tire is rolling without braking, the friction force is simply rolling friction and the coefficient of rolling friction between tires and a dry paved surface varies between 0.12 and 0.30

8.4.5 Braking Torque

The application of brakes supplies a torque to the -wheel which retards wheel rotation, however, initially, the retarding torque is balanced by the increase in frictional force which produces a driving or rolling torque. Figure 2.3.

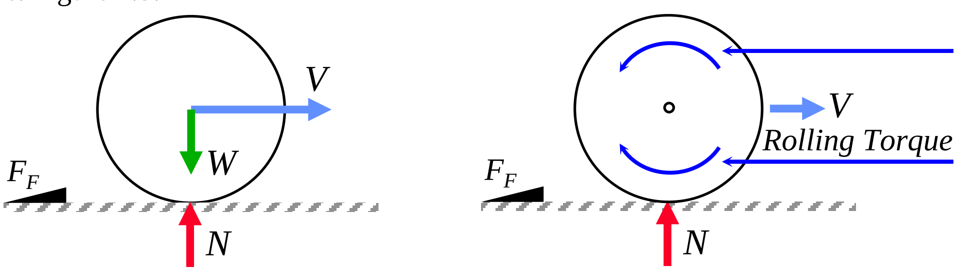


Figure 2. 3 The Relationship between Friction Force F_F, and Normal Force, *N* Braking Torque and Rolling Torque.

When the braking torque is equal to the rolling torque, the wheel maintains a constant angular velocity with no deceleration. Thus the application of brakes develops a retarding torque and an increase in frictional force between the wheel and the surface. An increase in brake pressure increases the braking torque to a value greater than the rolling torque and an apparent slip occurs between the wheel and the surface.

The effect of slip velocity on the coefficient of friction is illustrated in Figure 2.4. Zero slip corresponds to the locked wheel where the relative velocity between the tire and the surface equals the actual linear velocity *V*. The application of brakes increases the coefficient of friction as the tire incurs a small but measurable apparent slip velocity. As the slip velocity increases, the coefficient of friction increases to a peak value, then decreases until at the 100 % slip condition on dry concrete, then the coefficient of friction is decreased approximately 35 % from the peak value at about 8 % slip. The peak value of μ occurs at an incipient skid condition and the relative slip consists primarily of elastic shearing deflection of the tire structure.

Tire composition can have a considerable effect on the peak value of μ for dry surface condition. For example, a soft gum rubber tire can develop very high values of μ but only for low values of surface shearing stress, such as wet surface conditions. At high values of surface shearing stress the soft gum rubber tire will shear or scrub off before high values of μ are developed.

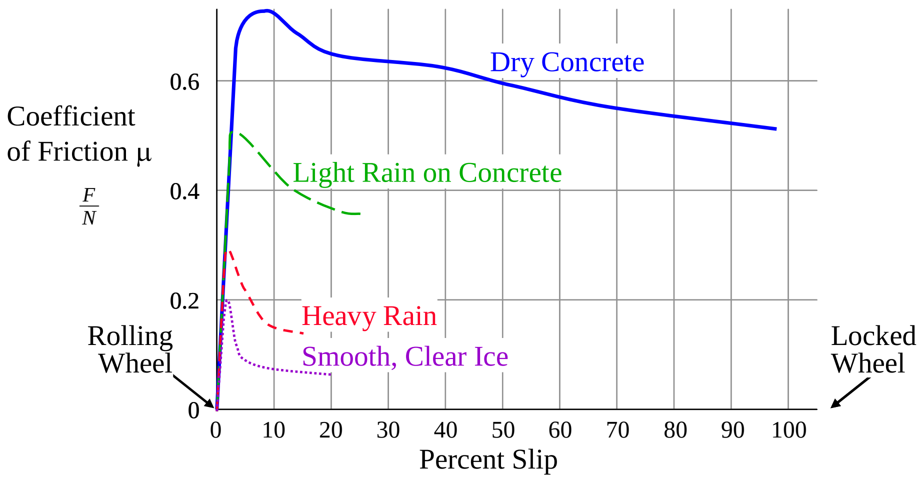


Figure 2. 4 Effect of Slip on the Coefficient of Friction for Various Types of Surface Conditions

If high traction on dry surfaces is the single design condition, then the optimum tire would be a soft rubber tire of extreme width to create a large footprint and reduce surface shearing stresses as per the tires on drag race cars. However, such tires have high rolling friction, large size and poor side force characteristics which makes them undesirable for normal operation.

When the runways or road surfaces are wet, the tread design is important to maintain contact between the surface and the tire and to prevent a film of water from lubricating the surface which can cause hydroplaning.

Figure 2.4 clearly shows that once the slip velocity exceeds the incipient skidding condition, the coefficient of friction decreases with the rate of decrease getting larger for wet and icy runway conditions. Once a skid begins, the reduction in friction force and rolling torque must be countered with a reduction in braking torque otherwise the wheel will decelerate and lock. If wheel locking occurs, the retarding force is reduced and the tires become incapable of developing any significant side force which can result in a loss of directional control of the vehicle. Therefore, if skidding occurs due to an excessive wheel slip condition, the driver must release the brakes to maintain or regain directional control, then reapply the brakes at a reduced pressure to continue to stop the vehicle. Aircraft are susceptible to skidding, particularly on wet and icy runways. However the larger aircraft tend to have anti-skid systems which sense slip velocities and automatically reduce the brake pressure to prevent skidding and wheel lock conditions.

Consider an aircraft that weighs 650,000 lb landing on a dry runway, capable of a maximum coefficient of friction of 0.75. The friction force generated by the brakes to stop the aircraft is easily determined as follows:

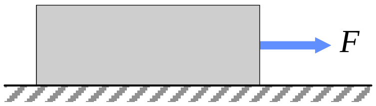
$$F_F = \mu R = 0.75\,(650{,}000) \text{ assuming that the wing lift is zero.}$$

$$= 487{,}500 \text{ lb.}$$

The above value of retarding force is significantly reduced on a wet or icy runway to 130,000 lb if the coefficient of friction is 0.2.

8.4.6 Tutorials

1. The block dimensions are 2 ft. high with a base of 4 ft. by 4 ft. The density of the material is 3.5 slug/ft^3.



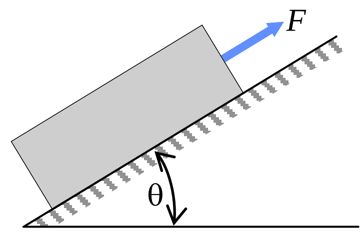
Find:

a) the weight of the block

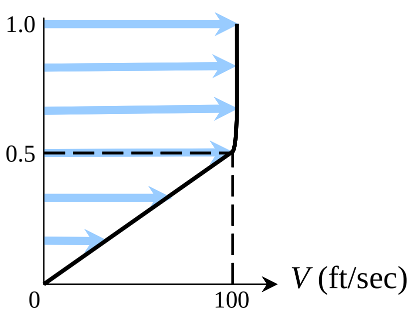
b) the minimum force F required to move the block horizontally if the coefficient of friction $\mu = 0.25$

c) the minimum force required to move the block if the block was turned on its side. (height = 4 ft).

2. A block is on an inclined plane as shown and weighs 150 lb. Find the minimum and maximum force to hold the block at rest on the plane if the coefficient of friction is 0.3 and 9 = 30 degrees.



3. The mean velocity boundary layer profile on a flat plate was measured-as shown.



a) find the velocity gradient dV/dy

b) find the shearing stress on one side of the flat plate (5 ft × 4 ft) if $\mu = 1.2024 \times 10^{-5}$ lb sec/ft

4. An aircraft weighs 230,000 lb. accelerates to V_1 speed (145 kts. TRUE) when an engine fails and the captain immediately reduces thrust and applies the brakes. Assuming that the average lift of the wings is 100,000 lb throughout the deceleration and the anti-skid system can keep the percent slip at 8 %, using the data from Figure 2.4 (dry concrete)
   a) Determine the average retarding force on the aircraft
   b) Determine the average retarding force in light rain with % slip = 4.0 %
   c) Determine the average retarding force in heavy rain with % slip = 2.0 %

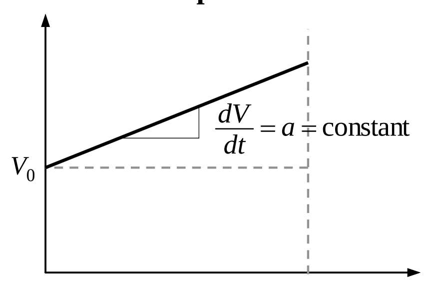
8.5 Kinematics

8.5.1 Introduction

Kinematics is a study of motion without regard to the forces or other factors that initiate the motion. In this chapter, translation and rotational motion will be studied with vibrational motion being introduced in a later chapter. Translation motion is sometimes called rectilinear motion and essentially studies motion that operates under constant acceleration, e.g. gravity. Development of the equations will be performed graphically and mathematically.

8.5.2 Linear Motion

Graphical



where V = final velocity

V_0 = initial velocity

$$V = V_0 + \frac{dv}{dt}t = V_0 + at$$

$$\boxed{V = V_0 + at}$$

Distance traveled during the time interval t is the area under the curve which is equal to S.

S = area of the rectangle + area of the triangle

$$S = V_0 t + \tfrac{1}{2}at(V - V_0)$$

but $$V - V_0 = at$$

$$\boxed{S = V_0 t + \tfrac{1}{2}at^2}$$

To determine a relation between distances and velocity V, requires substituting for time $t = \frac{V - V_0}{a}$ into the above equation.

$$S = V_0 \frac{V - V_0}{a} + \tfrac{1}{2}a\left(\frac{V - V_0}{a}\right)^2$$

$$S = \frac{V^2 - V_0^2}{2a}$$

or $$\boxed{V^2 = V_0^2 + 2aS}$$

Mathematical

Acceleration a = $\frac{dV}{dt}$ = constant

$$\int_{V_0}^{V} dV = a\int_0^t dt$$

$$V - V_0 = at$$

or $$\boxed{V = V_0 + at}$$

Velocity = $\frac{\Delta S}{\Delta t} = \frac{dS}{dt} = V_0 + at$

$$\int_{l_0}^{S} dS = \int_0^t (V_0 + at)dt$$

$$S \boxed{= V_0 t + \tfrac{1}{2}at^2}$$

To find a relationship between velocity, acceleration and distance:

$$a = \frac{dV}{dt} = \frac{dV}{dS}\cdot\frac{dS}{dt} = V\frac{dV}{dS} = \text{constant}$$

therefore, $a = V\frac{dV}{dS}$

$$\int_{V_0}^{V} VdV = a\int_0^S dS$$

$$\left[\tfrac{1}{2}V^2\right]_{V_0}^{V} = aS$$

$$\tfrac{1}{2}\left(V^2 - V_0^2\right) = aS$$

which leads us to the same solution as the graphical solution ;

$$\boxed{V^2 = V_0^2 + 2aS}$$

The above equations can be used to solve any linear problem in kinematics which has a constant acceleration.

Example: A car is traveling at 75-mph. and is stopped in 250 ft. using brakes. Assuming a constant deceleration, determine;

a) The deceleration due to the brakes, use the equation: $V^2 = V_0^2 + 2aS$

$$0 = [75(1.47)]^2 + 2a(250)$$

Note: 75 mph is changed to ft/sec (1 mph = 1.47 ft/sec)

Therefore, $a = -24.31$ ft/sec^2

b) The time to bring the car to a stop use the equation: $V = V_0 + at$

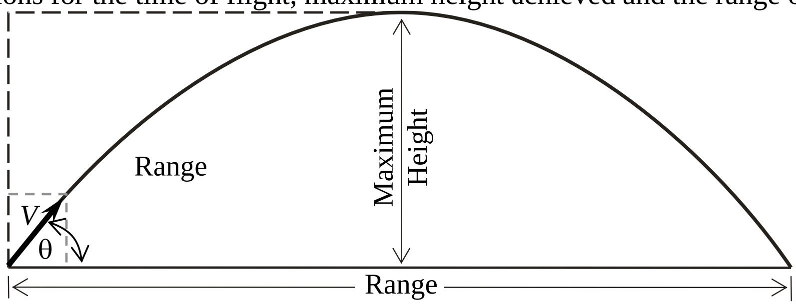
$$0 = (75)(1.47) + (-24.31)t$$

$$t = 4.535 \text{ secs.}$$

8.5.3 Curvilinear Motion

Curvilinear motion is a combination of both translation and rotational motions and applies to the trajectories of ballistic bombs and missiles etc...

Assume that a projectile is fired at an angle of θ to the horizon and has an initial velocity of V ft/sec. Determine expressions for the time of flight, maximum height achieved and the range of the projectile.

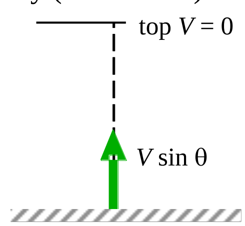


The basic approach is to split the initial velocity into two components:

horizontal velocity (V_H) = $V \cos\theta$ which is not affected by the acceleration due to gravity

vertical velocity (V_V) = $V \sin\theta$ which is acted upon by the acceleration due to gravity (32.2 ft/sec^3)

Consider the vertical component of velocity $V \sin\theta$. The velocity will decrease to zero at the maximum height achieved, reverse direction and accelerate back down to earth and hit the start point at the same velocity $V \sin\theta$, assuming no losses.



Final velocity at the top = $0 = V \sin\theta - a\,t$

$$t = \left[\frac{V \sin\theta}{32.2}\right] \text{sec}$$

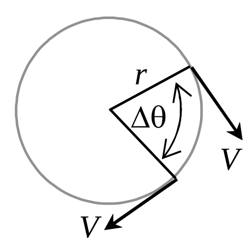
Therefore total flight time = $2\,t = \left[\frac{2V \sin\theta}{32.2}\right]$ sec

The maximum height can be found using the equation $V^2 = V_0^2 + 2aS$

Since $V = 0$, $V_0 = V \sin\theta$, a = –32.2 ft /sec^2 $\quad S = \frac{(V \sin\theta)^2}{2(32.2)}\ ft$

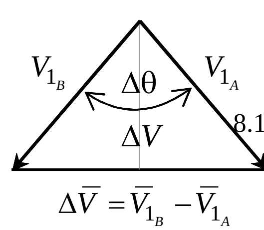
The projectile range = (Horizontal velocity component) (Time of Flight) =

$$(V\cos\theta)\left(\frac{2V\sin\theta}{32.2}\right) ft \;=\; \frac{V^2 \sin 2\theta}{y}$$



8.5.4 Circular Motion

In rectilinear motion, the velocity vector was constant in direction and only the magnitude changed. In circular motion, the velocity vector remains constant in magnitude; however, the direction continually changes.



Consider the velocity vector rotating around a circle of radius *r* ft. as shown, over an angle displacement of $\Delta\theta$. The velocity vector change is the vector sum of V_{1_A} and V_{1_B}.

The acceleration vector is $\frac{dV}{dt}$. As Δt approaches zero, $\Delta\theta$ approaches zero. Then, the instantaneous acceleration to the velocity vector approaches 90° to the velocity vector. This acceleration towards the center is called the centripetal acceleration and is required to produce curved flight paths.

The radial acceleration, is $$a_r = \frac{dV}{dt}$$

but $\Delta V = V_1 \sin \Delta\theta$ and for very small angles $\Delta V = V_1\, \Delta\theta$.

Therefore, $$a_r = \frac{V_1 \Delta\theta}{\Delta t} = V_1 \frac{\Delta\theta}{dt} = V_1 \omega$$

where $\omega = \frac{d\theta}{dt}$ = angular velocity in rad/sec

V_1.= tangential velocity

r = radius

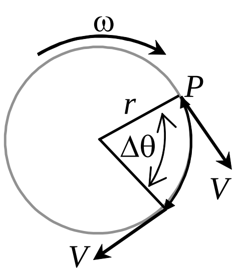
Also, the velocity $V_1 = \omega\, r$

Therefore $$a_r = (\omega r)\omega = \omega^2 r$$

Centripetal Acceleration = $\omega^2 r$

Therefore, for a particle to be following a circular flight path, it must have an acceleration towards the center of rotation of magnitude ωr^2.

8.5.5 General Relationships



Consider a circle of radius *r* and a point *P* moves around the side at a constant tangential velocity of *V*.

Then, $\theta = \frac{S}{r}$ or $s = r\theta$

where r = the radius in ft.

s = the arc of the circle subtended by θ in ft.

θ = the arc angle in radians.

Note: A 360° circle = 2 π radians, 1 radian = 57.3 degrees

The *angular velocity* ω in radians/second is the change in θ with respect to time

$$\omega = \frac{d\theta}{dt}$$

and the *angular acceleration* $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

The *tangential velocity V* at any point on the circle of radius *r* which is moving at an angular velocity ω is ω*r* in ft/sec.

$$V = \omega r \text{ or } \omega = \frac{V}{r}$$

The *tangential acceleration* a ft/sec at any point an the circle of radius *r* which is accelerating at an angular acceleration α is α*r* ft/sec

$$a = \alpha r \text{ or } \alpha = \frac{a}{r}$$

Also, by analogy with the linear kinematic equations;

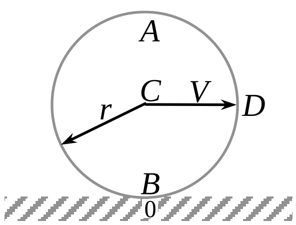
$$\omega = \omega_o + \alpha t$$

$$\theta = \omega_o t + \tfrac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_o^2 + 2\alpha\theta$$

8.5.6 Example Problems

1 A wheel is rolling without slipping on a surface. The instantaneous velocity of the center of the wheel is *V* ft/sec. Find the tangential velocity of point *A*, the angular velocity ω and the acceleration of point *A*.



Note: If the wheel is rolling without slipping then the instantaneous velocity of *B* with respect to the surface is $V_{B\text{-surface}} = 0 = V_B = 0$.

The horizontal velocity of the wheel center '*C*' relative to the surface is *V*

$$V_{C\text{-}0} = V$$

Then the instantaneous angular velocity of point *C* relative to the surface 0

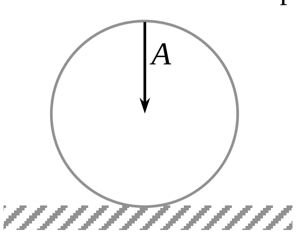
$$\omega_{C-0} = \frac{V}{r} \text{ or } \omega_{B-C} = \frac{V}{r} \text{ or } \omega_{A-C} = \frac{V}{r}$$

Therefore, the wheel is turning at a constant angular velocity ω =*V*/*r* rad/sec.

The absolute value of the tangential velocity of A relative to the surface = the tangential velocity of *A* relative to the center of the wheel plus the velocity of the wheel center *C* relative to the surface.

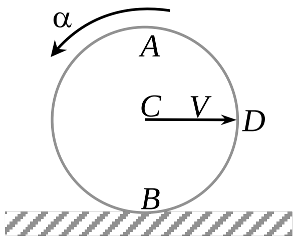
$$\begin{aligned}\vec{V}_A &= \vec{V}_{A-C} + \vec{V}_{C-0}\\ &= \omega r + V\\ &= V + V\\ \vec{V}_A &= 2V\end{aligned}$$

The acceleration of point *A* is towards the center of rotation and equals $\omega^2 r$.



$$a_{A-0} = \omega^2 r = V\omega = \frac{V^2}{r}\left(\frac{\text{ft}}{\text{sec}^2}\right)\downarrow$$

2. Assume that the wheel in example 1 had an angular acceleration α rad/sec^2 applied in the direction shown. Find the acceleration of point *A* and of point *D*.

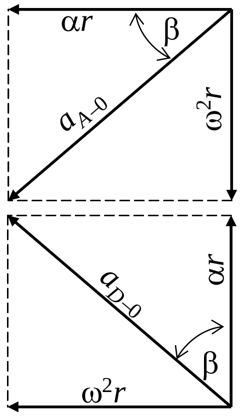


The acceleration of point *A* consists of two parts:

a) Acceleration towards the center of the wheel.
b) Acceleration due to the angular acceleration $\bar{\alpha}$.

$$a_{A-C} = a_{A-C} + a_{C-0} + a_{cent.}$$
$$= \overleftarrow{\alpha r} + 0 + \omega^2 r \downarrow$$

The acceleration of point *D* consists of the acceleration towards center *C* and the linear acceleration due to α.



$$a_{A-0} = \sqrt{\left(\alpha r^2\right)^2 + \left(\omega^2 r^2\right)}$$
$$\tan\beta = \frac{\omega^2 r}{\alpha r}$$

$$a_{D-0} = a_{D-C} + a_{C-0} + a_{cent.}$$
$$= \alpha r \uparrow + 0 + \overleftarrow{\omega^2 r}$$
$$a_{D-0} = \sqrt{(\alpha r)^2 + \left(\omega^2 r\right)^2}$$
$$\tan\beta = \frac{\omega^2 r}{\alpha r}$$

8.5.7 Tutorials

1. An aircraft touches down at 140 Kts. (Note: 1 Kt.= 6080 ft/hr) on a runway. The average deceleration is 4.5 ft/sec^2. Find:
   a. the total distance to stop [6,220 ft]
   b. the runway remaining when the aircraft was still doing 60 kts. The runway is 10,000 ft. long. [4932 ft]

2. A bullet is projected vertically with an initial velocity of 950 ft/sec. Neglecting drag, determine:
   a. the time of flight [59.0 sec]
   b. the maximum height reached. [14,014ft]

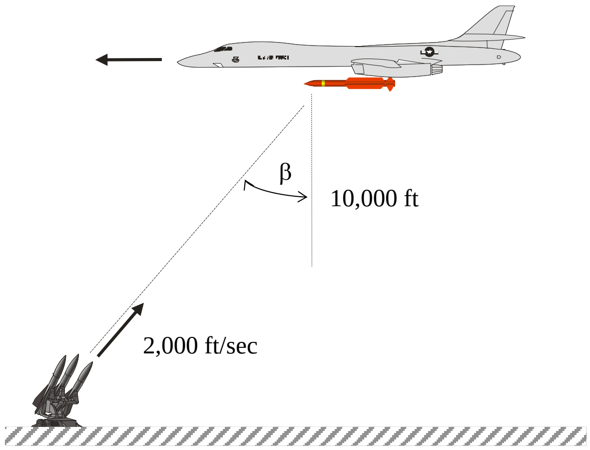
3. A point moves along a straight line. It uniformly accelerates from rest to 48 ft/sec to the right in 2 sec. The acceleration is then changed to a different constant value such that the displacement for the entire period is 48 ft. to the right and the total distance traveled is 192 ft. Determine:
   a. the total time interval
   b. the final velocity.

4. The position of a particle moving along a horizontal line is given by the equation
   $$S = 4t^3 - 21t^2 + 18t - 4$$
   where S is the distance from the origin in ft. and t is the time in seconds. The particle is 3 ft to the right of the origin when t = 1 sec. Determine :
   a. the times when the velocity is zero [$t = \frac{1}{2}$,3]
   b. the acceleration when t = 2 sec. [+ 6 ft/sec^2]
   c. the distance traveled during the interval from t = 0 to t = 4 sees. [54.5 ft]

5. An elevator (lift) starts moving upward at t = 0 when 10 ft. above the ground and the vertical velocity is constant at 2 ft/sec. Also at t = 0 a ball is projected vertically upwards at 50 ft/sec from a position 40 ft. above the ground. Determine:
   a. at what time will the ball hit the elevator [3. 33 sec]
   b. the height of the elevator above the ground at impact [26.7 ft]
   c. impact velocity of ball and elevator [62.9 ft/sec]

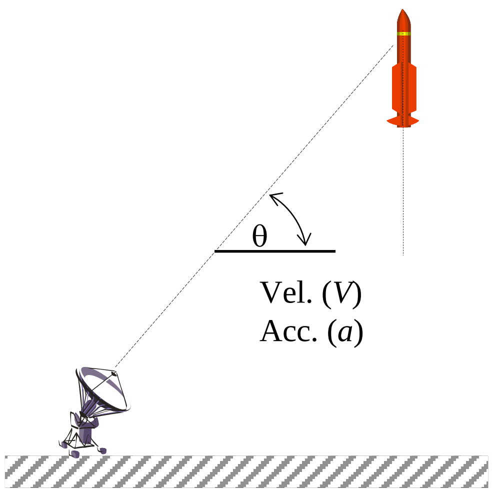
6. A ballistic missile is fired at an angle of 30° to the horizontal with a muzzle velocity of 3500 ft/sec. Neglecting drag, determine;
   a. the missile range
   b. the height of the apogee
   c. the time of flight.
7. A fighter aircraft is overtaking a bomber at the same altitude. The aircraft are flying at 800 ft/sec and 600 ft/sec respectively. The fighter fires a missile when 1900 ft. behind the bomber. The missile accelerates at 1000 ft/sec^2 for 1 sec. then travels at a constant speed. Determine:
   a. the time for the missile to reach the bomber
   b. the distance between aircraft -when the missile strikes (no change in velocity)

8. An aircraft flying level at 10,000 ft. A.G.L. releases a bomb to hit a gun site.



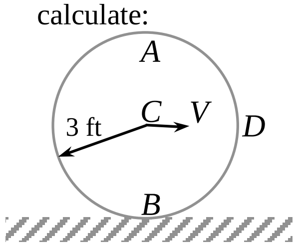
a. determine the angle β of the bomb site on release to hit the gun site.
b. determine the velocity for which β = 45°
c. where will the bomber be when the bomb impacts?
d. if the gun can launch a ballistic projectile at 2000 ft/sec, what lead angle is required at the bomb release angle at 10,000 ft for both the bomber and the gun to be destroyed?

9. A radar is tracking a rocket missile from a location '*d*' ft from the launch site.



a. determine the velocity of the rocket in terms of *d*, θ , *d*θ/*dt*
b. the acceleration of the rocket in terms of *d*, θ , $\frac{d\theta}{dat}, \frac{d^2\theta}{dt^2}$

10. A car driving around a flat circular track at 60 mph covers a distance of $\frac{1}{4}$ mile per circuit. Compute the angular velocity in rad/sec and the centripetal acceleration.
11. A mass of 32.2 slugs is rotating at 90 degrees per second on a length of string 3 ft. long. Determine the tangential velocity, the centripetal acceleration and the tension on the string. Assume *force* = (*mass*) (*acceleration*)
12. A turbine wheel rotates at 35,000 RPM and is 2.5 ft. in diameter. If each blade weighs 3 oz., calculate the angular velocity in rad/sec, the centripetal acceleration, the centrifugal force in the blade and the velocity of the blade if it left the turbine wheel.
13. The wheel shown has a radius of 3 ft and is moving to the right at $V = 5$ ft/sec with a linear acceleration of point *C* to the right of 3 ft/sec. Assuming the wheel is rolling without slipping, calculate:



a. the velocity of point *A*
b. the acceleration of point *A*
c. the acceleration of point *D*
d. the acceleration of point *D* assuming the ground surface is accelerating to the left at 1 ft/sec^2.

8.6 Forces and Moments on Motion

8.6.1 Introduction

Kinematics involved the study of motion without regard to the forces or moments that caused the motion. In this chapter, we shall study the laws that govern the effect of forces and moments on motion.

8.6.2 Mass and the Conservation of Mass

Mass is defined as a measure of the quantity of matter in a body and is independent of the gravitational field acting upon the body; however, the weight of a body is dependent upon the gravitational field acting on the body. For example, a body on the moon will weigh only about 18 % of what it weighs on the Earth. Similarly, a body outside a gravitational field will be weightless, but still have mass.

In a strict sense, mass is a function of velocity but until space-travel velocities increase significantly, i.e., approach the speed of light, it can be assumed that mass is constant.

The law of conservation of mass states that mass can be neither be created nor destroyed. However, mass can be converted into energy, for example, fuel. An application of this law is that all mass in a given system can be accounted for.

A mass will resist any change in motion either in magnitude or direction and this resistance is called inertia. This resistance to change is the basis for Newton's First Law of Motion.

8.6.3 Newton's First Law of Motion

"A body at rest tends to remain at rest, and a body in motion tends to remain in motion in a straight line unless acted upon by an external force."

Thus, an external force (engine thrust) must be applied to an aircraft to accelerate the aircraft on the runway during the take-off ground roll. The external forces and their effects on a body form the basis of Newton's Second Law.

8.6.4 Newton's Second Law

"If a body is acted on by an external force, the body will accelerate in the direction of the force and the acceleration is directly proportional to the external force and inversely proportional to the mass of the body.

$$acceleration = \frac{force}{mass}$$

or $F = M \times a$ is the common form of Newton's Second Law.

If the force has the units of lb and acceleration the units of ft/sec^2, then mass has the units of lb sec^2/ft which is assigned the simplifying term of "slug". One slug mass is accelerated one ft/sec^2 for every pound of force applied.

The application of Newton's Second Law is the foundation for the study of linear aircraft dynamics.

In the study of rotary motion and rotary dynamics of aircraft, the basis for these equations is derived from a modification of Newton's Second Law which states that an external moment acting on a body will cause that body to accelerate in an angular manner about an axis according to the equation:

Moment = (Moment of inertia about the axis of rotation) (Angular acceleration)

$$Moment = I. \times a$$

where I is the moment of Inertia (mass ft^2) and a is the angular acceleration (rad/sec).

The moment of inertia is the resistance to angular motion of a mass.

8.6.5 Newton's Third Law

"To every action there is an equal and opposite reaction". This law is seen quite clearly when firing a rifle; the force required to prefect the bullet is reacted by the shooter and is called recoil. Similarly, the force required to accelerate air through an engine

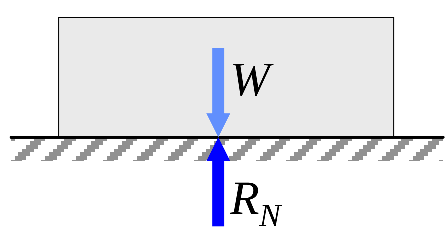


Figure 4.1 Reaction between Body and Surface

or propeller or rotor to generate a rate of change of momentum is reacted by a thrust from the engine, propeller or rotor. A body at rest on a table has a weight W towards the center of the earth; however, this weight is reacted by the normal reaction between the body and the surface as shown in Figure 4.1

Another example of action and reaction is centripetal and centrifugal force. Centripetal force is the force which provides the radial acceleration to turn a body in a curved flight path. Centripetal force acts inwardly towards the center of rotation. Centrifugal force is the reaction to the centripetal force.

If a mass (M) is attached to a string and is rotating in a circle of radius (r) at a constant angular velocity ω rad/sec, then. using Newton's Second Law we can calculate the centripetal force towards the center.

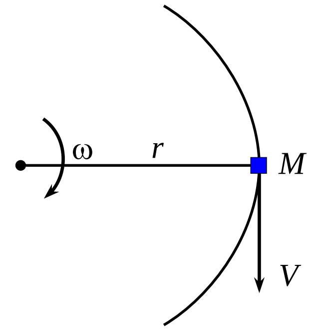


Figure 4. 2 Rotating Mass on a String

$$F = M \times a$$

Centripetal force = (mass) (acceleration towards the center)

$$Centripetal\ force = m \times \left[\omega^2 r\right] = \left[\frac{W}{g}\right]\left[\omega^2 r\right] = \left[\frac{W}{g}\right]\left[\frac{V^2}{r}\right]$$

where V = tangential velocity (ft/sec) = $\omega\ r$ and r = radius of rotation (ft)

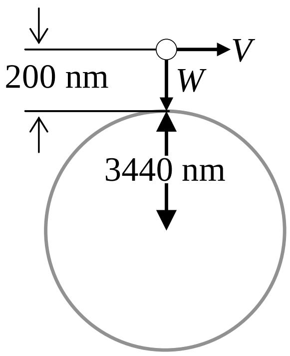


Figure 4.3

An example of centripetal force is an earth satellite in circular orbit at an altitude of 200 nautical miles as shown in Figure 4.3. If the radius of the earth is 3440 nautical miles and the acceleration due to gravity is 28.6 ft/sec^2 at 200 nm, then the tangential velocity required to maintain the prescribed orbit can be easily determined from the above equations.

$$Centripetal\ force = \left[\frac{W}{g}\right]\left[\frac{V^2}{r}\right] = W$$

since it is the weight vector towards the center of the earth that is the force that is accelerating the mass towards the center of the earth and generating a curved flight path.

Therefore, $V^2 = g\ r$

where V = orbital velocity (ft/sec)

g = acceleration due to gravity (ft/sec)

r = radius from center of earth to satellite (ft)

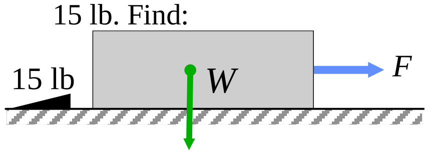
Therefore, V^2 = (28. 6) (3440 + 200) (1.689) = 634,000,000 ft^2/sec^2

and V = 25.200 ft/sec = 14,900 kts.

The laws relating force, mass and acceleration for linear motion and moments, moments of inertia and angular acceleration for rotary motion are used to set-up and solve the equations of motion for all dynamic responses.

8.6.6 Examples of the Applications of Newton's Laws

1. A force is applied to a block on a surface as shown. The block weighs 100 lb. and the friction force is 15 lb. Find:



a. the initial acceleration if the force F is 45 lb.
b. the distance traveled in 5 sees.
c. the velocity after traveling 10 ft. assuming that the block starts from rest.

Newton's Law - Out of balance Force = Mass (acceleration)

a. $$(45 - 15) = \left[\frac{W}{g}\right] \times a$$

$$a = \frac{(32.2)(30)}{100} \text{ ft / sec}^2$$

$$a = 9.66 \text{ ft / sec}^2$$

Since the force applied to the block and the frictional force are constant, then the acceleration will be constant and the basic laws of Kinematics can be applied.

b. $\Delta t = 5$ seconds and initial velocity = 0

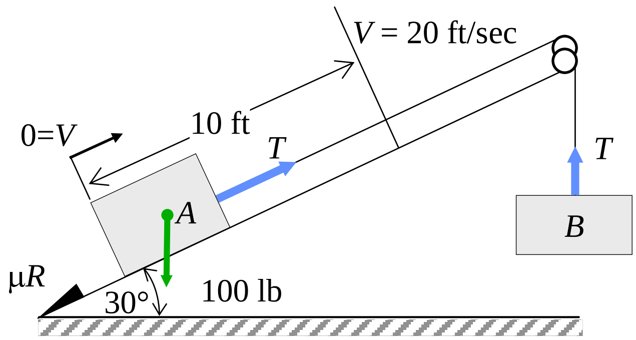
using the equation:

$$S = V_0 t + \tfrac{1}{2} a t^2$$
$$= 0 + \tfrac{1}{2}(9.66)(5)^2$$
$$= 120.75 \text{ ft}$$

c. $S = 10$ ft. $V_0 = 0$, find the velocity after 10 ft.

$$V^2 = V_0^2 + 2aS$$
$$= 0 + 2(9.66)(10)$$
$$= 13.9 \text{ ft / sec}^2$$

2. Block A weighs 100 lb and starts from rest and travels up the incline and reaches a velocity of 5 ft/sec after 10 ft. Assume μ = 0.05 for the block sliding on the incline. Determine the weight of block B to achieve this motion. Neglect the moment of inertia of the pulley.



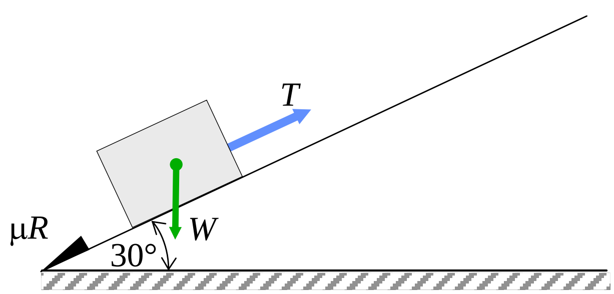
Step 1: Consider block *A* only and that the tension in the cord pulling the block up the incline is equal to *T*.

Step 2: Apply the basic Kinematic equation to determine the acceleration of the block *A* up the incline:

initial velocity	$V_0 = 0$
final velocity	$V = 5$ ft/sec
distance traveled	$S = 10$ ft

$$V^2 = V_0^2 + 2aS$$
$$5^2 = 2(a)10$$
$$a = \left[\frac{25}{20}\right] \text{ft / sec}^2$$

Step 3: Set up the free body diagram for block *A*. Analyze the forces on the block to find the value of the tension in the cord that will give the acceleration of $\left[\frac{25}{20}\right] \text{ft / sec}^2$.



Resolve the force parallel to the plane:

$T - \mu R - W \sin 30$ = force up the plane.

This force will accelerate the block A up the plane.

$R = W \cos 30 = (100)\ (0.\ 86603) = 86.6$ lb.

$$T - (0.05)(86.6) - (100)(0.5) = F = \frac{W}{g}a$$

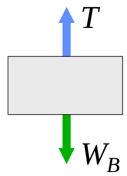
$$T - 4.33 - 50 = \left[\frac{100}{32.2}\right]\left[\frac{25}{20}\right]$$

$$T = \left[\frac{100}{32.2}\right]\left[\frac{25}{20}\right] + 54.33$$

$$T = 58.21 \text{ lb} \quad \text{Tension in the cord.}$$

Step 4: The tension in the cord will be a constant throughout the cord. Therefore, set up the free body diagram for the block *B* knowing the cord tension *T* and the acceleration of block *B* which must be the same as block *A*.

The out of balance force of block *B* is



$$W_B - T = \frac{W_B}{g}a$$

$$W_B - 58.21 = \left[\frac{W_B}{32.2}\right]\left[\frac{25}{20}\right]$$

$$W_B(1 - 0.03882) = 58.21$$

$$W_B = 60.56 \text{ lb}$$

3. An aircraft is flying at 300 kts. true airspeed when full aileron deflection is applied. The mass moment of inertia about the roll axis is 18,000 slugs ft^2 and the initial roll acceleration was measured as 2 rad/sec^2. Find the moment on the aircraft about the roll axis.

Moment = (Moment of Inertia) (Angular Acceleration)

M = 18000 x 2

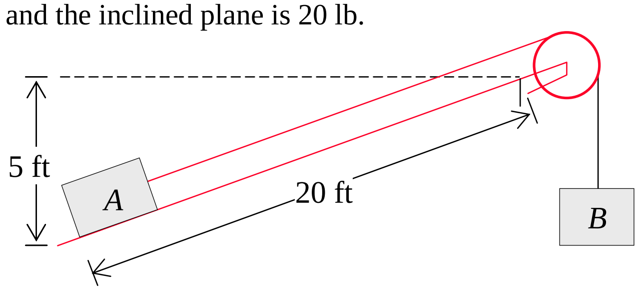
M = 36000 lbs ft

8.6.7 Tutorials

1. A 50,000 lb missile is launched vertically. After a time interval, the thrust produced by the rocket motor is 500,000 lb and the aerodynamic drag is 5,000 lb. Find the acceleration of the missile.

[285 ft/sec^2]

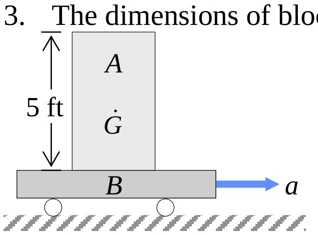
2. Block *A* is pulled up the incline by block *B*. Block *A* weighs 500 lb and the frictional force between *A* and the inclined plane is 20 lb.



Determine the weight of *B* that will cause *A* to reach the top of the incline with a velocity of 10 ft/sec starting from rest at the bottom of the incline. Neglect the mass of the pulley.

[199.3 lb]

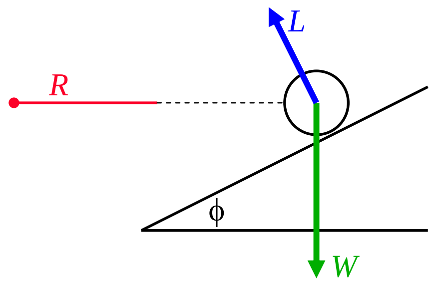
3. The dimensions of block *A* are 3 ft by 3 ft by 5 ft and the weight of the block is 1200 lb.



The block rests on a carriage *B* which is given an acceleration (*a*) in the direction shown. If the friction between the block and the carriage is sufficient to prevent slipping, what is the maximum acceleration that the carriage can have without causing the block to topple over?

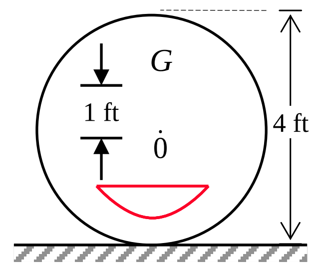
[19.32 ft/sec]

4. An aircraft is making a constant speed horizontal turn. Find the load factor '*n*' as a function of the bank angle and the radius of turn as a function of '*n*' and the turn rate ω.



5. The wheel weighs 322 lb and has a radius of gyration of mass of 1.5 ft with respect to an axis through the center of mass '*G*'. A 225 ft lb couple *T* acts on the wheel. The coefficient of static friction between the horizontal surface and the wheel is 0. 40. Determine the magnitude of the maximum angular velocity the wheel can have in the position shown if the wheel does not slip.

[ω = 4.15 rad/sec]



8.7 Momentum and Impulse

8.7.1 Introduction.

Momentum is defined as the product of mass and the true linear velocity, i.e., momentum = (mass) (true velocity) = *mV* slug ft/sec, therefore, momentum is a vector quantity. For example, the momentum of a 25,000 lb aircraft touching down on an aircraft carrier deck at 110 kts true velocity is;

$$\left[\frac{25{,}000}{32.2}\right](110 \times 1.689) = 144{,}332.3 \text{ slug ft/sec}$$

Angular momentum is defined as the product of the moment of inertia and the angular velocity in radians per second.

$$(\text{Angular Momentum}) = I\,\omega$$

8.7.2 Conservation of Momentum

Momentum will be conserved unless acted upon by external forces or moments, i.e.,

$$m_1 V_1 = m_2 V_2 \quad \text{or} \quad I_1 \omega_1 = I_2 \omega_2 \quad \text{etc...}$$

A classical example of conservation of angular momentum is the pirouetting ice skater who starts a pirouette with arms and leg outstretched as shown below, at a low angular velocity ω_1. To speed up the angular motion, the skater brings her arms and leg in towards the spin axis thereby reducing the moment of inertia and since

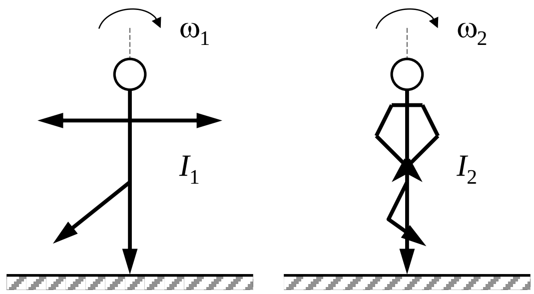


Figure 5.1 Sketches of the Ice Skater in a Pirouette

$$\omega_1 I_1 = \omega_2 I_2$$

by the conservation of momentum theory, the ω_2 must increase to compensate for the decrease in I_2. To slow down again, the skater extends her arms which increases the moment of inertia thereby slowing the angular velocity.

Another example of the conservation of angular momentum that sometimes takes inexperienced pilots by surprise when performing spins in trainer aircraft is:

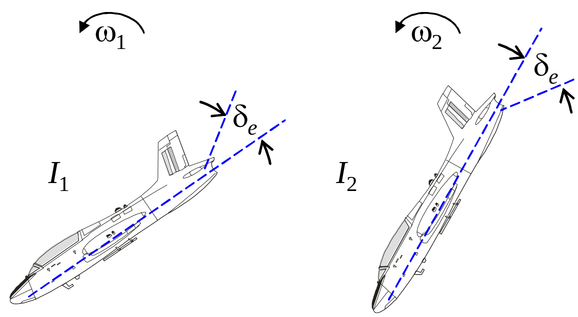


Figure 5.2 Spin Recovery in a Training Type Aircraft

when down elevator is applied, the nose pitches down, which decreases the moment of inertia I_2 about the spin axis and actually increases the spin rate of the aircraft initially.

Arrester cables situated at the end of runways sometimes consist of a cable stretched across the runway with each end attached to a large heavy chain lying on the ground as shown in Figure 5.3

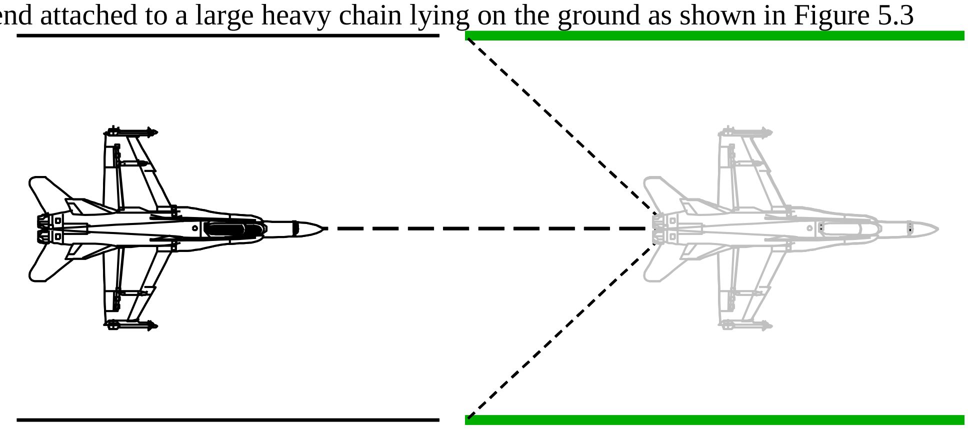


Figure 5.3 Arrestor Cable at End of Runway

As the aircraft picks up the cable with its arrestor hook. it starts picking up more and more mass of the heavy chains which increases the mass of the combined A/C and cable and chain, thereby reducing the velocity. The progressive increase in mass as the aircraft travels between the chains as shown reduces the initial deceleration to a safe structural level for the aircraft and the pilot.

8.7.3 Coefficient of Restitution

The principle of conservation of momentum assumes no losses: however, in the collision of bodies, the energy absorbed by the bodies in deformation, heat, etc... is a function of the elastic properties of the bodies. Thus, the ratio of the velocities of the bodies that collide with direct normal impact, to their relative velocity of approach is defined as the coefficient of restitution for the two bodies. The coefficient of restitution can vary from 0 to 1.0 depending upon the elastic properties of the bodies.

$$\text{Coefficient of Restitution} = e\frac{\left[\frac{V_a}{B}\right]_f}{\left[\frac{V_a}{B}\right]_i}$$

When the impact of the two bodies is oblique, i.e., not normal, the components of the velocities of the points of contact normal to the surface of contact are used.

8.7.4 Linear Impulse

The linear impulse of a force which varies in magnitude but not in direction during a time interval is:

$$\text{Impulse } = \int_{t_i}^{t_f} F dt$$

where t_i and t_f. are the initial and final values of the time. Linear impulse is a vector quantity with the same sense as the force and the dimensions are in (lb sec).

The principle of linear impulse and linear momentum can be derived from Newton's laws of motion and are expressed as:

" The linear impulse of a force system acting on a body during a time interval is equal to the change in linear momentum of the body during the same time interval. "

$$[\text{change in impulse}]_0^t = [\text{change in momentum}]_0^t$$

Since the change in force over a time interval is a change in momentum, then

$$\Delta \text{ Force = rate of change of momentum = } \frac{d}{dt}(m \cdot V)$$

since impulse is the product of force and the time interval over which it acts. If a constant force is assumed, then the impulse can be expressed as

$$\text{Impulse} = F\,(t_2 - t_1) = J$$

For example, if a rocket engine produces a constant thrust of 500,000 lb for 50 seconds, the total impulse of the engine would be

$$\text{Impulse = (500,000) (50) = 25,000,000 lb sec.}$$

Impulse is used directly to rate the output of rocket engines and often the term 'specific impulse' is used. Specific impulse (*SI*) is a measure of the amount of thrust which can be obtained from each pound of propellant in one second of engine operation. It can also be considered as the total impulse divided by the weight of the fuel expended.

$$\text{Specific Impulse} = \frac{\text{Total Impulse}}{\text{Total weight of fuel consumed}}$$

$$SI = \frac{J}{W_f}$$

however, $$J = T(t_2 - t_1)$$

therefore, $$SI = \frac{T(t_2 - t_1)}{W_f}$$

or $$SI = \frac{\frac{T}{W_f}}{(t_2 - t_1)} = \frac{T}{\dot{W}_f}$$

where $\dot{W}_f$ is the fuel consumption in (lb fuel/sec).

Note: Precise units of specific impulse are pounds of thrust per pound of fuel per second and since the pound units cancel, the final units for specific impulse (*SI*) are seconds.

Example: If a rocket engine produces 500,000 lb thrust for 50 seconds and consumes 125,000 lb of fuel, the specific impulse (*SI*) is

$$SI = \frac{500{,}000}{125{,}000/50} = 200\,\text{lb/lb/sec} = 200\,\text{sec}$$

Since thrust is the force which imparts impulse to a missile, the effect of the impulse on the momentum of the missile becomes evident. The change in momentum of the missile equals the impulse of the force (thrust) exerted on the missile.

8.7.5 Angular Impulse

The moment of the linear impulse of a force about any axis is called the angular impulse of the force with respect to the given axis.

$$\text{Angular Impulse} = \int_{t_i}^{t_f} (Moment)\,dt$$

or, if the applied moment is constant, then the

Angular Impulse = (Moment) (time moment is applied)

$$A.I. = I\,\omega$$

8.7.6 Angular Momentum

The angular momentum is defined as:

Angular Momentum = (Moment of Inertia) (Angular Velocity)

$$A.M. = I\,\omega$$

8.7.7 Principle of Angular Impulse and Angular Momentum

The principle states that the sum of the moments about an axis through the mass center of the body is equal to the rate of change of the angular momentum of the body with respect to the axis.

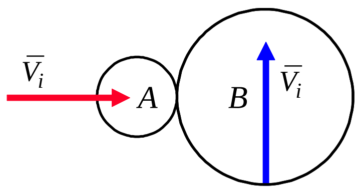
$$\text{Angular Impulse} = \int_{t_i}^{t_f} (Moments)\,dt = I(\omega_f - \omega_i)$$

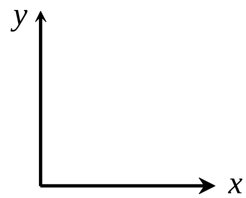
where ω_f and ω_i are the final, and initial values of the angular velocities at times t_f and t_i respectively. This equation is valid only if the axis is through the mass center or the axis of rotation.

In summary, the angular impulse of the forces acting on any body with respect to an axis through the mass center of the body is equal to the change in angular momentum of the body with respect to the same axis.

8.7.8 Sample Problems

1. Disks *A* and *B* slide on a smooth horizontal plane and collide with oblique central impact. Disk *A* weighs 10 lb and has a velocity of 20 ft/sec, to the right before impact. Disk *B* weighs 20 lb. and has a velocity of 12 ft/sec upward before impact. The disks are smooth and have a coefficient of restitution of 0.5. Determine:





   a. the velocity of *B* after impact
   b. the change in kinetic energy of *B* due to impact.

$$\Delta LM_x = 0$$

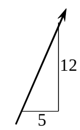
$$\frac{10}{g}(10) + 0 = \frac{10}{g} V_{A_{f_x}} + \frac{20}{g} V_{B_x}$$

$$0.5 = \frac{V_{A_{f_x}} - V_{B_{f_x}}}{10 - 0}$$

Solving $V_{B_x} = 5 \text{ ft/sec} \rightarrow$

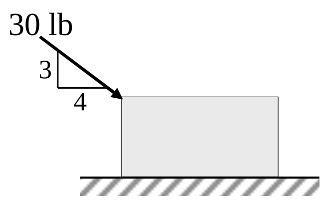
$V_{B_y} = 12 \text{ ft/sec} \uparrow$ (no change in y)

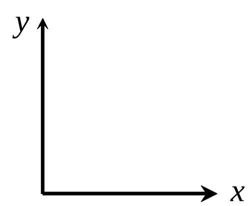
$V_{B_f} = 13 \text{ ft/sec}$



$$\Delta E_k = \tfrac{1}{2} \frac{20}{g} \left[13^2 - 12^2\right] = 7.76 \text{ ft - lb gain}$$

2. The coefficient of friction between the 64.4 lb block and the plane is 0.20. How long does it take for the 30 lb force to change the velocity from 12 ft/sec to the left to 18 ft/sec to the right?





$$N = 64.4 + \frac{3}{5}(30) = 82.41 \text{ lb}$$

$$F = M \times N = 16.48 \text{ lb}$$

Impulse = change in momentum

$$LI_x = \Delta LM$$

$$(24 + 16.48)\Delta t_1 = 2[0 - (-12)]$$

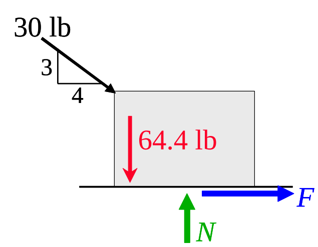
$$\Delta t_1 = 0.593 \text{ sec}$$

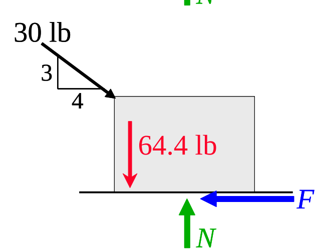
$$LI_x = \Delta LM$$

$$(24 - 16.48)\Delta t_2 = \frac{64.4}{32.2}[18 - 0]$$

$$\Delta t_2 = 4.78 \text{ sec}$$

$$\Delta t = 5.37 \text{ sec}$$





3. If a Jet engine powered aircraft has a true flight velocity of 900 ft/sec (532 kts) and the engine air mass flow rate is 20 slugs per second and the exhaust velocity is 1200 ft/sec determine the thrust produced by the engine.

$$\text{Thrust = Rate of change of momentum} = \frac{d}{dt}(m \times V) = \frac{m}{t}(V_{ex} - V_{in})$$

where m/t = air mass flow rate, slugs/sec.
V_{ex} = exhaust gas velocity (ft/sec)
V_{in} = inlet gas velocity (ft/sec)

therefore the thrust = 20 (1200 – 900) = 20 (300) = 6000 lb

4. In rocket engines the exhaust gases are produced by the burning of the fuel in the rocket and the gas inlet velocity is zero.

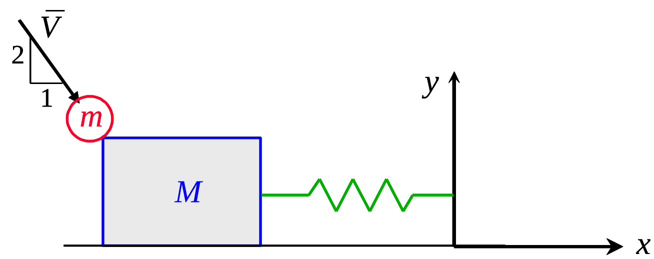
Therefore, $$\text{the thrust} = \frac{d}{dt}(m \times V) = \frac{m}{t}(V_{exhaust})$$

Assuming that a rocket bums 1000 slugs/sec and the exhaust velocity is 2500 ft/sec. the thrust produced is:

thrust = (1000) (1500) = 1.500,000 lb.

5. A spring-mass system resting on smooth surface is struck by a mass, *m*, which has a velocity $\bar{V} = 5\bar{i} - 10\bar{j}$ ft/sec. The mass '*m*' adheres to the spring-mass system which has a mass '*M*'.

a. what is the velocity of the two masses immediately after impact?
b. what impulse does the floor exert on the system assuming the velocity after impact is all in the '*x*' direction?



Impulse in the '*y*' direction does not change. Therefore,

a. $LI_y = 10m$

b.
$$LI_x = \Delta LM$$
$$L(0) = \Delta LM$$
$$m(5) + M(0) = (m + M)V_{f_x}$$
$$V_{f_x} = \frac{5m}{(m + M)}$$

6. A small mass '*m*' is attached to a string and rests on a smooth table top. The string passes through a hole in the table top and is held in position by a force *T*. If the initial length of the string from the hole to the mass is *L* and the mass is traveling in a circular arc at a speed *V*, how fast is the mass traveling if the string length is shortened by an amount *b*?

Conservation of angular momentum:

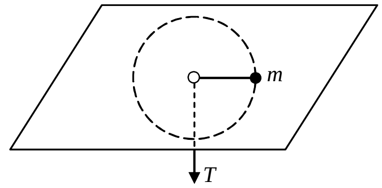
$$I_i\omega_i = I_f\omega_f$$

$$I_i = mL^2$$

$$I_f = m(L-b)^2$$

$$\omega_i = \frac{V}{L}$$

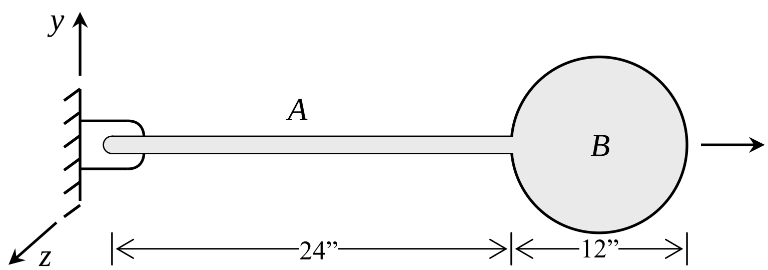
$$\omega_f = \frac{V_f}{L-b}$$



$$\left(mL^2\right)\frac{V}{L} = m(l-b)^2\frac{V_f}{L-b}$$

$$V_f = \frac{L}{L-b}V$$

7. The homogeneous uniform 16.1 lb bar *A* and the 64. 4 lb homogeneous cylindrical disk *B* are welded together to form a rigid body that is symmetrical with respect to the *x*,*y* plane. The body rotates about the *z* axis with an angular velocity of $-15\bar{k}$ rad per second in the position shown.
   a. determine the angular momentum of the body with respect to the origin 0.
   b. locate the linear momentum vector of the body.



$$I_0 = \tfrac{1}{3}(0.5)2^2 + \tfrac{1}{2}(2)(0.5)^2 + 2(2.5)^2 = 13.42$$

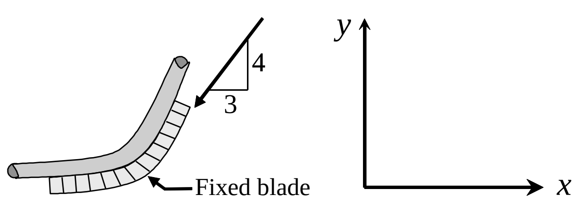
a. $A\,M_0 = I_0\,\omega$ = 13.42 (15) = 201 ft-lb-sec

b. $A\,M = r\,(L\,M)$

$$201 = x\left[\tfrac{1}{2}(1)15 + 2(2.5)(15)\right]$$

$$x = 2.44 \text{ ft}$$

8. A cylindrical jet of water 2" in diameter impinges on a fixed blade. The velocity of the jet is 20 ft/sec, to the right. Determine the force exerted on the blade by the water. The blade is smooth.



Since the blade is smooth $\frac{V_j}{B}$ = constant.

$$\left[\frac{\bar{V}_j}{B}\right]_i = \left[\bar{V}_j - \bar{V}_B\right]_{\bar{i}} = 20_{\bar{i}} - 0 = 20_{\bar{i}}$$

$$\left[\frac{\bar{V}_j}{B}\right]_f = 20\left[\frac{3_{\bar{i}} + 4_{\bar{i}}}{5}\right]$$

$$\left[V_j\right]_f = \left[\frac{V_j}{B} + V_{oB}\right]_f = 20\left[0.61\bar{i} + 0.8\bar{j}\right] = 12\bar{i} + 16\bar{j}$$

$$\left[V_j\right]_f = \left[\frac{V_j}{B} + 0\right]_f = 20\left[0.61\bar{i} + 0.8\bar{i}\right] = 12\bar{i} + 16\bar{j}$$

$$\left[V_j\right]_f = \left[\frac{V_j}{B}\right]_f = 20\left[0.61\bar{i} + 0.8\bar{i}\right] = 12\bar{i} + 16\bar{j}$$

$$\overline{R}\Delta t = m\left[\overline{V}_{jf} - \overline{V}_{ji}\right]$$

and $$m = \frac{2\pi}{12}\left[V_{jB}\Delta t\right]\rho$$

$$\overline{R}\Delta t = \left[\frac{2\pi}{12}\right]^{-2}\frac{\pi}{4}\left[20\Delta t\right]\rho\left[12\bar{i} + 16\bar{j} - 20\bar{i}\right]$$

$$-\overline{F} = \overline{R} = -15.02\left[0.448\bar{i} - 0.89\bar{j}\right]$$

8.8 Work and Energy

8.8.1 Linear Motion.

Work is defined as a force multiplied by the distance traveled by the force. If the force has the units of pounds (lb) and the distance measured in feet (ft) then the units of work are (ft lb). Note that work does not include the mass of a body unless the body is being- raised vertically as shown in Figure 6.1 and the force moved through a distance x ft is the mass of the body (m) times the acceleration due to gravity (g) (ft/sec) and the work done to move the mass is ($W\,x$) or ($m\,g\,x$) ft lb.

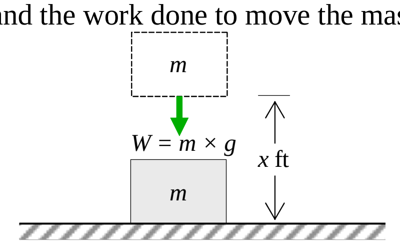


Figure 6.1 Work $= W\,x = m\,g\,x$ ft lb

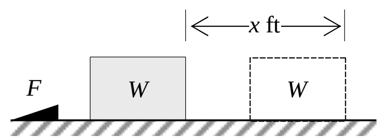


Figure 6.2 Work $= F\,x$ ft lb

If a body is resting on a flat plane as shown in Figure 6.2 and the frictional force between the block and the surface is F lb, then the work required to move the block through a distance of 'x' ft is ($F\,x$) ft lb.

8.8.2 Angular Motion

In an angular system, with a force applied to a wheel as shown in Figure 6.3, the force F lb applies a torque ($F\,r$) (ft lb) about the center of the wheel 0. If this force moves the wheel through an angular displacement θ radians, then the work done is the torque times the angular displacement.

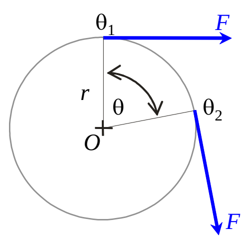


Figure 6.3 Work in Angular Motion

$$\begin{aligned} \text{work done} &= \text{torque} \times \text{angular displacement} \\ &= F \cdot r \text{ (ft lb)} \times \theta \text{ (rads)} \\ &= F \cdot r \cdot \theta \text{ ft lb} \end{aligned}$$

8.8.3 Energy

Doing work on a system can increase the total energy of the system. For example, the work done on the block 'm' in Fig. 6.1 is $W\,x$ (ft lb) and this work done on the block raises the block 'x' ft above the surface plane. By virtue of its position above the plane, the block has the potential to do work if released from its position 'x' above the plane, and if there were no losses due to aerodynamic drag etc.., then the block could do work equal to ($W\,x$) (ft lb) when falling from its position 'x' ft above the surface back to the surface. The ability to do work due to a position above a reference plane is called potential energy and is defined as the •weight of the body (W) (lb) times its vertical distance above a reference plane in ft.

$$\text{Potential Energy} = W\,x \text{ (ft lb)}$$

Obviously then, work done on a body or system of bodies can increase its potential energy. This is not always true since in Figure 6.2 the work done to move the block 'W' from position A to position B did not change the potential energy of the block. However, by sliding the block over the surface and overcoming the frictional force 'F' generated heat between the block and the surface. This heat is another form of energy; therefore, a more correct statement would be that doing work on a body or system of bodies increases the total energy level of the body or system of bodies.

8.8.4 Types of Energy

Potential Energy: (PE) The energy due to the position of a body above a reference plane.

$PE = (Weight)\,(Height)$ (ft lb)

$PE = W \cdot H$ (ft lb)

Kinetic Energy: (KE) The energy contained in a body of mass 'm' moving at a velocity V ft/sec.

$KE = \frac{1}{2} m \cdot V^2$ and since mass = $\frac{weight}{g}$

$KE = \frac{1}{2}\frac{W}{g} \cdot V^2$ (ft lb)

Checking the units: $\frac{W(\text{lb})}{g(\text{ft/sec}^2)} \cdot V^2$ (ft/sec)^2 = ft.lb

Rotational Energy: (RE) The energy contained in a body of mass (m) rotating about a center of rotation with a moment of inertia I and an angular velocity of ω radians/sec.

$RE = \frac{1}{2} I \cdot \omega^2$ ft.lb

Checking the units

$$I\left(m \cdot ft^2\right) \times \frac{1}{\sec^2} = \frac{m}{\left(\text{ft} / \sec^2\right)} \cdot \text{ft}$$

$$= \frac{m}{g} \cdot \text{ft}$$

$$= W \cdot \text{ft} = \text{ft} \cdot \text{lb}$$

Vibrational Energy: (VE) The energy contained in a body vibrating in a linear or rotary type motion.

Heat Energy: (HE) The energy contained in a mass due to its temperature above its surrounding environment.

Electrical Energy: (EE) The energy due to the voltage difference between two parts of a body.

Chemical Energy: (CE) The energy stored in a substance that can be extracted by burning the substance in oxygen. Petroleum is an example of chemical energy.

8.8.5 Energy Storage

The storage of energy for use at a later date is important for many physical applications and many methods exist such as batteries for storing electrical energy, fuel for storing heat energy, rotating masses for storing rotational energy, such as a flywheel, water towers for storing potential energy and springs for storing energy when compressed or extended. Energy storage techniques essentially consist of converting one form of energy into another form which can be stored for a short time with minimum losses; however, most energy storage systems involve losses and have a limited life. An energy storage system that is of interest to the storage of energy is a linear or rotary spring which is often used to drive turning mechanisms. A linear spring of spring stiffness 'K' lb/ft can store energy when the spring is stretched a distance of 'x' ft. The energy stored in the spring is the area under the spring characteristics curve shown in Figure 6.4 and is $\frac{1}{2}(x)(Kx) = \frac{1}{2}Kx^2$ ft lb

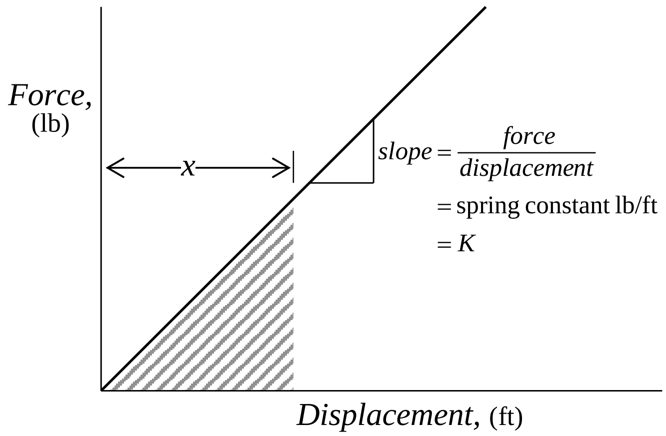


Figure 6.4 Characteristics of a Linear Spring

8.8.6 Total Energy

The total energy of a body consists of the sum of all the energies listed in section 6.4.

Total Energy = Potential + Kinetic + Rotational + Vibrational + Heat + Electrical + etc.

However, in flight testing the only energies that are of interest are the first three.

$$TE = PE + KE + RE$$

The rotational energy (*RE*) is extremely important in helicopter applications where the stored rotational energy of the rotor blades is used to stop the rate of sink and the forward speed of a helicopter in autorotational power off landings

In fixed wing applications the rotational energy of the engine or the propellers is very small in comparison with the potential and kinetic energy and is usually neglected. Therefore,

$$\underset{TE}{Total\,Energy} = \underset{PE}{Potential\,Energy} = \underset{KE}{Kinetic\,Energy}$$

8.8.7 Specific Energy Height

Another form of energy is used in energy performance studies -which assume that the weight of the aircraft is constant and does not change with the result that the total energy can be divided by the weight of the aircraft resulting in the total energy per lb of aircraft which is named *specific energy*. Specific energy has the units of ft.

$$Specific\ Energy = H + \tfrac{1}{2}\frac{V^2}{g}\ \text{ft} = h_e\ \text{ft}$$

Lines of constant specific energy height can therefore be constructed as shown in Figure 6.5 using any combinations of absolute height (*H*) above sea level and the true velocity of the aircraft in ft/sec.

The use of the specific energy map, Figure 6.5, is very useful in determining the total energy of an aircraft by virtue of its height and velocity and if it is assumed that energy can be interchanged without losses as is usually the case in energy performance studies, then the potential zoom altitude of an aircraft can be estimated. For example, if an aircraft is flying at point *A* in Figure 6.5, at 20,000 ft and 1600 ft/sec, true velocity, then by using the assumptions of no losses when interchanging potential and kinetic energy, the aircraft flying at point *A* has the ability (theoretically) to zoom to 50,000 ft at zero forward velocity or to dive into the ground at 1900 ft/sec. This is accomplished by moving up or down the line of constant specific energy height.

The total energy level of an aircraft is very important in air to air combat and the aircraft that arrives at the intercept with the most total energy has the better chance of winning the initial engagement. The aircraft with the most total energy can make the decision, either to engage or not or to dive through the target, shoot and leave without danger of being overtaken by the target aircraft with the lower energy level. The best that the lower energy target aircraft can hope to achieve is to see the aggressor aircraft attacking and have the time to turn into the aggressor and to have a nose to nose shoot out.

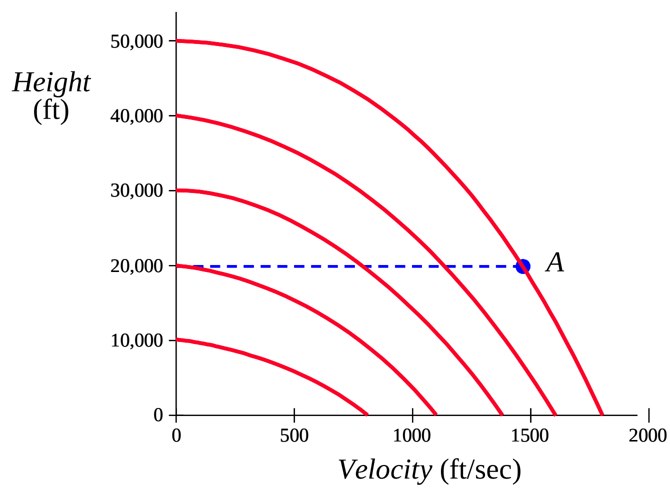


Figure 6. 5 Lines of Constant Specific Height

The low speeds of the aircraft of World War I and World War II meant that potential energy was a high percentage of the total energy; therefore, fighters flew as high as possible prior to engaging the enemy and preferably up-sun from the enemy to ensure surprise in the initial attack. A comparison of three typical aircraft from 1920, 1940 and 1980 is shown below. The total energy level of each type is given as well as the specific energy levels.

		W (lb)	V_{max} (kts)	*Height* (ft)	*PE*	*KE*	*TE*	Energy Height *PE*	Energy Height *KE*	Energy Height *TE*
A	1918 WW I	2000	150	20,000	40×10^6 (95 %)	1.99×10^6 (5%)	41.99×10^6 (100%)	20,000 (95%)	1,002 (5 %)	21,002 (100 %)
B	1945 WW II	8000	350	40,000	320×10^6 (88 %)	43.4×10^6 (12 %)	363.4×10^6 (100 %)	40,000 (88 %)	5,442 (12 %)	45.442 (100 %)
C	1985	40,000	1200	60,000	2.48×10^9 (48 %)	2.55×10^9 (52 %)	5.03×10^9 (100 %)	60,000 (48 %)	63,863 (52 %)	123,863 (100 %)

Table 6.1 Typical Filter Aircraft Energy Levels

A survey of the three aircraft in the above table shows the usefulness of the specific energy concept as it reduces the magnitude of the numbers and gives the maximum specific energy height ability of each aircraft making a comparison of the aircraft possible. The three aircraft are plotted on the energy height curves of Figure 6.7. The important point from this comparison is that the current supersonic fighter aircraft have as much energy due to their velocity as they do from their altitude and this makes energy performance techniques for the current fighter aircraft very interesting.

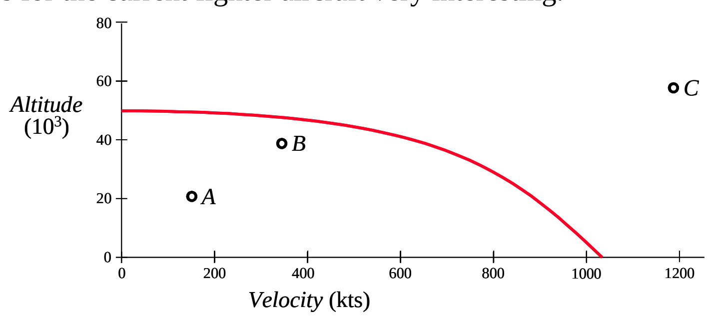
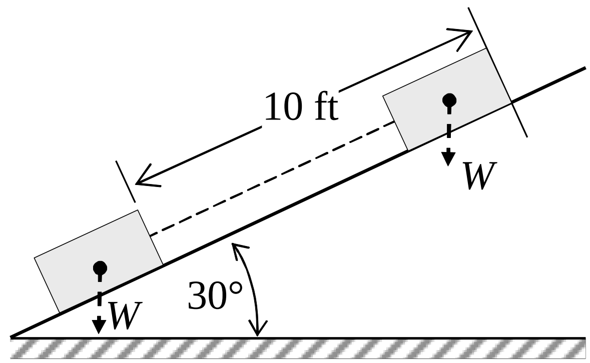


Figure 6. 7 Energy Height Curves

8.8.8 Tutorials

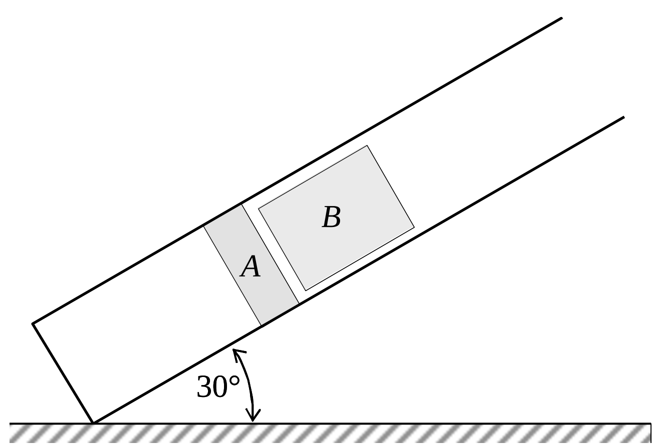


1. Determine the amount of work performed on a body which is moved 10 ft. up the incline as shown. Assume no friction between the block and the plane. The block weighs 150 lb.
2. Determine the amount of work performed on the body in question 1 if the coefficient of friction between the block and the plane is 0.20.
3. How much work is required to move a nut, 0.60 inches along a threaded screw with a 4 inch spanner if the force applied to the spanner is a constant 5 lb and the screw thread is 0.12 inches?
4. What is the potential energy of an aircraft flying at 36,000 ft altitude at a true velocity of 380 kts. The aircraft weighs 240,000 lb?
5. Determine the kinetic energy of the aircraft in question 4.
6. Determine the total energy of the aircraft in question 4 and what percentage of the energy is kinetic. Assume that the rotational energy is zero.
7. A fighter aircraft is flying at 45,000 ft. at a true airspeed of 1200 Kts. when the engine is shut down. The aircraft is zoomed to a maximum altitude where the true velocity is 350 Kts. Assuming a 10 % loss of energy during the zoom climb, calculate the maximum altitude of the aircraft.
8. A twin engine aircraft has two propellers each of 8 ft in diameter. The propellers each weigh 450 lb and have a radius of gyration of 2 ft. Determine the rotational energy of the aircraft propellers when they are running at 2000 RPM.
9. An 8000 lb helicopter is flying at 5000 ft at a forward speed, of 90 kts. The main rotor weighs 1000 lb and has a radius of gyration of 20 ft. The rotor is operating at a constant 300 RPM.

a. find the total energy of the helicopter and

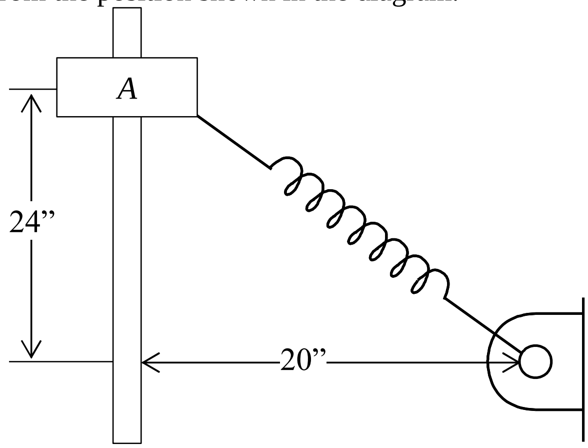
b. the percentage of the total energy contained in the main rotor.

10. A spring cannon has a spring of unstretched length of 3 ft. and at the top of the spring is attached a 20 lb base plate A. The bottom of the spring is attached to the base of the cannon. The spring constant is 300 lb/inch. If a 50 lb projectile (*B*) is loaded into the cannon as shown and the spring is compressed six inches prior to release,
    a. find the velocity of the projectile when the base plate is about to separate from the projectile.
    b. assuming no losses due to friction or aerodynamic drag, estimate the maximum height of the apogee and the range of the projectile.



11. An aircraft is flying at 35,000 ft and at 1000 kts true airspeed. The aircraft weighs 35,000 lb.
    a. Find the specific energy height of the aircraft
    b. assuming no losses when interchanging potential and kinetic energy, find the maximum velocity of the aircraft at sea level.
12. A collar of 20 lb weight slides freely on a vertical metal rod as shown. If the unstretched spring length

is 26 inches and the spring constant is 10 lb/inch, find the lowest position the collar *A* would descend on the pole if released from the position shown in the diagram.



Volume 1 – Math & Physics for Flight Testers

Chapter 9

Thermodynamics Review

Table of Contents

9.1 Preface

The aerodynamics and propulsion courses in the NTPS syllabus make use of some of the basic relationships of thermodynamics. This chapter highlights the significant aspects of energy transfer that relate to flight testing. It begins with basic definitions of the continuum and the Equation of State. Next, the Law of Conservation of Mass and Newton's Laws of Motion are described. Finally, the First and Second Law of Thermodynamics are put forth.

A *continuum* defines the concept of continuous matter as distinguished from the kinetic theory or domain of molecular effects. Any gas is composed of a large number of molecules in continuous motion and collision. The most fundamental but cumbersome way of analyzing fluid motion would be to write and solve the equations of motion for every particle. Another approach is to treat the motion of a large number of molecules on a statistical basis. This is the kinetic theory of gases and has considerable merit, but is still too cumbersome or most practical calculations. For the class of problems which we deal with, the behavior of molecules is of little interest. We are concerned with the gross behavior of the fluid considered as continuous matter. Pressure, temperature, and density are important to us; molecular speeds are not.

A fluid is continuous when the smallest volume of fluid of interest contains so many molecules that average values are meaningful. Throughout this text, the parameters density, pressure, velocity, internal energy, enthalpy and entropy describe the gross behavior of a gas assumed to be continuous.

Consider the mass of gas m inside some volume V. At first the mass density *(m/V)* tends to be constant. If we choose an excessively small volume, then it contains only a few molecules. In this area the density varies with time as molecules enter and leave the volume. The smallest volume, which can be regarded as continuous then, is a practical limitation to our analysis.

9.2 Equations of State

The Equation of State relates the properties of a gas at any given state.

9.2.1 Perfect Gas Law

There is a functional relationship between pressure, density, and temperature such that if any two of the parameters are known, the value of the third can be calculated. This relationship changes somewhat with extreme low temperatures or high pressures (in which case we must use Van der Waals' Equation) but for the most part, the following **Perfect Gas Law** is appropriate.

$$P_v = RT \qquad (10.1)$$

or

$$P = \rho g R T$$

where

P = absolute pressure *(lb/ft²)*

ρ = mass density in *(slugs/ft³)*

g = earth gravitational constant (32.174 *ft/sec²*)

R = gas constant (*ft/Rankine)*

T = absolute temperature *(Rankine* = ℉ + 460)

v = specific volume = 1/g *(ft³/lbm)*

Editor's Note: Another version of the Perfect Gas Law, not used in this text, is $P=\rho R_1 T$ where R_1 equals gR. Always check units to be sure the equation matches the units of R).

As a matter of reference, in the Earth's atmosphere, sea level standard day ambient pressure, P_o = 14.7 *lb/in* =2116 lb/ft^2. Sea level standard ambient temperature T_o = 15°C = 288.15K = 59°F = 519R.

R = 53.3 *(ft/Rankine)* for air.

For gases other than air that obey the perfect gas law, the gas constant can calculated as

$$R = \left(\frac{1544}{molecular\ weight}\right)\left(\frac{ft-lb}{lb-{}^oR}\right)$$

9.2.2 Boyle and Charles' Law

Consider a gas at some state 1 for which $P_1v_1 = RT_1$. Some process takes the gas to state 2 for which $P_2v_2 = RT_2$. The following ratios between the gas properties come from the ratio of the perfect gas equation for each state.

$$\frac{P_1}{P_2} = \frac{T_1}{T_2}\frac{v_2}{v_1} \qquad \frac{T_1}{T_2} = \frac{P_1}{P_2}\frac{v_1}{v_2} \qquad \frac{v_1}{v_2} = \frac{T_1}{T_2}\frac{P_2}{P_1}$$

These equations are generalized statements of the laws of Boyle and Charles. **Boyle's Law** states that when the temperature of a given mass of gas is held constant, then the volume and pressure vary inversely. **Charles' Law** states that when the volume of a given mass of gas is held constant, then the change in pressure of the gas is proportional to the change in temperature.

9.2.3 Specific Heat

The specific heat of a substance *(C)* is the amount of heat required to raise the temperature of a certain mass by one degree. The *British Thermal Unit (BTU)* is defined as the amount of heat required to raise the temperature of one pound of water by one degree Fahrenheit. Logically enough, the specific heat of water is exactly 1 *BTU/lb°R.*

The specific heat analysis for gases is a little different since they can be heated in two different ways. First consider a constant pressure process. Do this experimentally by heating the gas within a cylinder that has a weighted piston above. The piston allows for expansion at a constant pressure. In this case, we calculate the amount of heat required to increase the temperature as

Heat required per pound = $\Delta T C_p$

Where C_p = specific heat at constant pressure $\left(\frac{BTU}{lb°R}\right)$

Similarly, the heating can take place in a fixed volume. In this case, the pressure will increase with heating and we calculate the amount of heat required to increase the temperature as

Heat required per pound = $\Delta T C_v$

Where C_v = specific heat at constant volume $\left(\frac{BTU}{lb°R}\right)$

The total heat *(Q)* required to change the temperature of an amount of substance is given by:

$Q = WC\Delta T$

Where W = weight of substance in pounds
C = specific heat; C_p or C_v

A need for the *ratio of specific heats* occurs frequently in practice and is defined as gamma (γ).

$$\gamma \equiv \frac{C_p}{C_v} \qquad (10.2)$$

This ratio is a key element in most aerodynamics analysis. For air, $\gamma = 1.4$ within the temperature region associated with aircraft flight.

9.2.4 Speed of Sound

The speed of sound is the speed at which a small pressure pulse will propagate though a medium. Although the derivation is outside the scope of this text, know that compressible fluids have a slower speed of propagation as seen by

$$a^2 = \left.\frac{\partial P}{\partial \rho}\right|_s$$

Where a is acoustic velocity. This partial derivative can be evaluated for a perfect gas as

$$a = \sqrt{\gamma g R T} \qquad (10.3)$$

Using the previously defined units for air give two convenient conversion factor:

$$a(knots) = 29.1\sqrt{T} \text{ (Rankine)}$$

$$a(ft/\text{sec}) = 49.1\sqrt{T} \text{ (Rankine}$$

The speed of sound at standard sea level temperature is

$$a_o = 1117\, ft/\text{sec} = 661 knots = 761 mph$$

9.2.5 Mach Number

Mach number is defined as the ratio of the local velocity to the local speed of sound. Flight Mach number is the aircraft velocity compared to the local speed of sound

$$M = \frac{V}{a} = \frac{V}{\sqrt{\gamma g R T}}$$

Mach number serves as a good index of the amount of compressibility the flow goes through. In approximate terms, flight testers break down aerodynamics into four regimes

$0 < M < .8$ subsonic
$.8 < M < 1.2$ transonic
$1.2 < M < 5$ supersonic
$5 < M < \infty$ hypersonic

9.3 Law of Conservation of Mass

This law gives an understanding of the mass flow rates through a system. With the exception of nuclear reactions, matter can neither be created nor destroyed, so the matter flowing into a system must be accounted for throughout. Define $\dot{m}$ as the mass rate of flow (*slugs/sec*) and G as the weight rate of flow (*lb/sec*). Mass flow is the product of density, volume and speed $\dot{m} = \rho A V_T$. Similarly, the weight rate of flow is $G = \rho g A V$ where

A = cross sectional area (ft^2)

V = average flow velocity (*ft/sec*)

Conservation of mass analysis must account mass flow going into and out of the system in question. Certain examples such as a streamtube, nozzle, diffuser, turbine, or compressor are especially straightforward since there is only one entrance and one exit. For these systems we use the continuity equation:

$$G_{ent} = G_{exit} = Constant$$

or

$$\dot{m}_{ent} = \dot{m}_{exit} = Constant$$

For certain low velocity conditions, the flow is essentially incompressible (*constant* ρ). For these conditions the continuity equation simplifies to the incompressible form:

$$AV = Constant$$

9.4 Newton's Laws of Motion

In thermodynamics Newton's classical laws of mechanics related fluid motion to the forces causing it. They are summarized as follows:

1. A particle at rest or traveling at constant velocity will tend remain at rest or at a constant velocity.
2. A particle acted upon by an unbalanced force will accelerate. The time rate of change of momentum is directly proportional to this unbalanced force:

$$Fex = \frac{d(mv)}{dt}$$

When mass is constant over the time period of interest, this equation reduces to the more familiar

$$F=ma$$

3. Every action has an equal and opposite reaction

These laws can be applied to a volume of fluid as well as a particle. Consider Newton's Law applied to flow through some passage (Figure 10.1). In practice this passage could be a streamtube around an airfoil, flow through a nozzle, a diffuser, or any other machine.

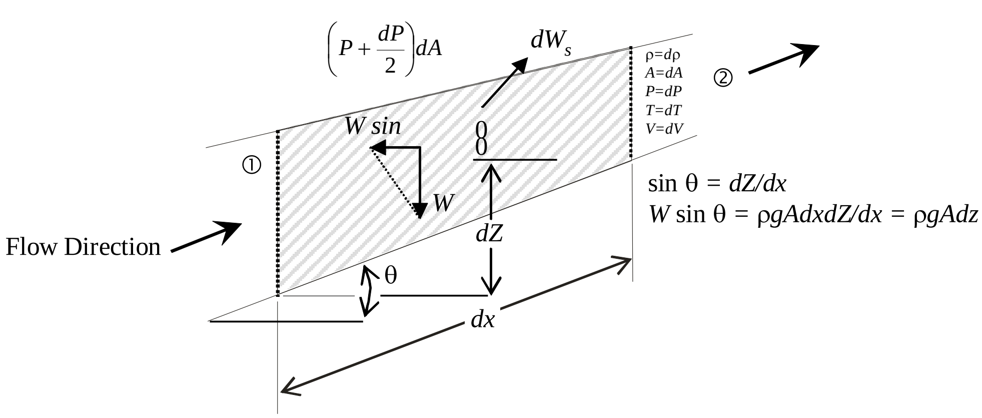


Figure 10.1 Flow Through a Passage

Restricting the development to one-dimensional flow, consider the four possible forces which can act on this chunk of matter and which act along the axis of flow:

a. Pressure forces acting on the front face (PA), on the rear face ($(P+dP)(A+dA)$), and on the sides $\left(P + \frac{dP}{2} dA\right)$
b. Weight forces $W \sin \theta = \rho g A dZ$ acting at the center of gravity.
c. Shaft of shear forces set up in a shaft which adds work to, or removes work from, a system (dW_s).
d. Friction forces along the side face (dW_f).

Neglecting dW_s and dW_f for the moment, sum the pressure and weight forces and apply Newton's 2nd Law to get:

$$+PA - (P + dP)(A + dA) + \left(P + \frac{dP}{2}\right) dA - \rho g A dZ = ma$$

or

$$+PA - PA - PdA - -AdP - dPdA + PdA + \frac{1}{2} dPdA - \rho g A dZ = ma$$

or

$$AdP - \frac{1}{2} dPdA - \rho g A dZ = ma$$

Note that $$m = \rho\left(A + \frac{dA}{2}\right) dx \text{ and } a = \frac{dV}{dt}$$

Inserting the mass and acceleration equivalents and neglecting higher order terms (products of very small quantities) gives:

$$-AdP - \rho A dZ = \rho A dx \frac{dV}{dt}$$

$$AdP + \rho g A dZ + \rho A dx \frac{dV}{dt} = 0$$

Divide through by $\rho A g$

$$\frac{dP}{\rho g} + dZ + \frac{dx}{g} \frac{dV}{dt} = 0$$

$$\frac{dP}{\rho} + dZ + \frac{1}{g} \frac{dx}{dt} dV$$

$$\frac{dP}{\rho} + dZ + \frac{V}{g} dV = 0$$

This is the differential form of the momentum equation. If shaft and/or friction work is present, the equation takes the form

$$\frac{dP}{\rho g} + dZ + \frac{VdV}{g} = -dW_s dW_f$$

We will neglect friction for the remainder of the review. The units of each term of the equation are ft-lb/lb or work per unit of fluid. To solve a practical problem we need to integrate this equation from state 1 to state 2 as follows:

$$\int_1^2 \frac{dP}{\rho g} + \int_1^2 dZ + \int_1^2 \frac{VdV}{g} = \int_1^2 - dW_s$$

Integrating gives **The Momentum Equation**

$$\int_1^2 \frac{dP}{\rho g} + \left(Z_2 - Z_1\right) + \frac{V_2^2 - V_1^2}{2g} = -W_{1-2} \tag{10.5}$$

Where for each unit (pound) of fluid:

$\int_1^2 \frac{dP}{\rho g}$ = Flow work done *on the system* between states 1 and 2.

$\frac{V_2^2 - V_1^2}{2g}$ = Change in kinetic energy between states 1 and 2.

$Z_2 - Z_1$ = Change in potential energy between states 1 and 2.

W_{1-2} = Shaft work done *on the flow* from state 1 and 2.

All of these terms except the first are easy to calculate. To integrate this term we must know how P and ρ vary for the process in question.

It is common to neglect the change in potential energy in aerodynamics problems (assume $Z_2 - Z_1 = 0$). Also, those flow processes in which no shaft work is applied to the flow are of special interest to us, since this applies to pitot statics, flow about an airfoil, in a nozzle, or an inlet. For $W_s = 0$, Equation (10.5) reduces to Bernoulli's equation for incompressible flow.

$$\frac{V_2^2 - V_1^2}{2g} + \frac{1}{\rho g}\left(P_2 - P_1\right) = 0\left[Incompressible\right] \tag{10.6}$$

With a constant ρ, then the flow work is $\frac{1}{\rho g}\left(P_2 - P_1\right)$.

9.5 Bernoulli's Compressible Equation

If ρ is not a constant it must be expressed in terms of P before we can integrate. This can be easy or difficult depending on the process. Most of our studies will be concerned with isentropic processes where P and ρ are related by Poisson's relation:

$$Pv^{\gamma} = \text{constant (for isentropic flow)} \tag{10.7}$$

Where $v = \frac{1}{\rho g}$

This relation will be developed later in this chapter.

Integrating *vdP* gives

$$\int v\,dP = \frac{\gamma}{\gamma - 1}\left[P_2 v_2 - P_1 v_1\right] \tag{10.8}$$

Applying this to Equation 10.5 gives

$$-W_{1-2} = \frac{V_2^2 - V_1^2}{2g} + \left(Z_2 - Z_1\right) + \frac{\gamma}{\gamma - 1}\left[\frac{P_2}{\rho_2 g} - \frac{P_1}{\rho_1 g}\right]$$

Again neglecting shaft work and change in potential energy, this equation reduces to Bernoulli's equation for compressible flow.

$$\boxed{\frac{V_2^2 - V_1^2}{2g} + \frac{\gamma}{\gamma - 1}\left[\frac{P_2}{\rho_2 g} - \frac{P_1}{\rho_1 g}\right] = 0[Compressible]} \tag{10.9}$$

Bernoulli's equations apply to flow processes where there is neither friction nor shaft work and the flow is either incompressible, or isentropic. An example of isentropic, frictionless flow with $W_s=0$ is flow up to a stagnation point. These relations form the basis of modern pitot-static theory. For flow through a nozzle, (Figure 10.2)

The Bernoulli equation also allow us to calculate fluid velocity at any point knowing only the pressure and density at the point in question and some reference state.

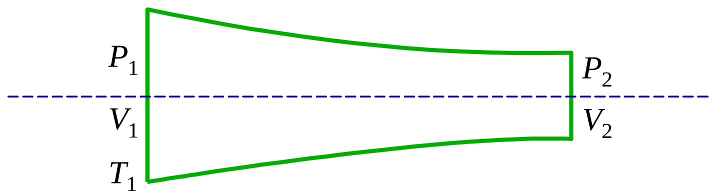


Figure 10.2 Nozzle Flow Schematic

9.6 Thrust Equation

Use Newton's Second Law to calculate the thrust from a jet engine, ramjet, rocket, propeller, or any other reaction device.

Considering steady flow:

$$\begin{aligned} F &= ma \\ &= (\dot{m}\Delta T)\left(\frac{\Delta V}{\Delta T}\right) \\ &= \dot{m}\Delta v \end{aligned}$$

Since $\dot{m}g$ equals G the thrust equation becomes

$$F = \frac{G}{g} dV \tag{10.10}$$

> Example: A jet aircraft is flying a Mach 1.0 at 36,000 feet on a standard day. The engine uses 85 pounds of air per second and the velocity of the exhaust gases is 1800 ft/sec. What is the thrust of the engine? (Ambient temperature = 395 R).
>
> 1. $a = \sqrt{\gamma g R T} = 49.1\sqrt{395} = 976\, ft/\text{sec}.$
> 2. $V=Ma$ = 976 *ft/sec*.
> 3. Every second the engine takes in 85 pounds of air at 976 ft/sec and discharges it at 1800 ft/sec.
> 4. $F = \frac{G}{g} dV = \frac{85}{32.2}(1800-976) = 2175 lb$

9.7 Law of Conservation of Energy

The First Law of Thermodynamics is the law of conservation of energy. This law states that energy can neither be created nor destroyed.

9.7.1 Energy-Work-Heat

Based on the First Law above the energy level of fluid can change only if energy is added to or subtracted from it. The First Law is commonly written that the change in energy equals the heat added to the fluid minus the work extracted from the fluid. Consider a process depicted in Figure 10.3.

$$E_2 - E_1 = Q_{1-2} - W_{1-2}$$

where Q_{1-2} = Heat added between 1 and 2
W_{1-2} = Work removed between 1 and 2
E_2 = Energy level of the flow at station 2
E_2 = Energy level of the flow at station 1

Before going any further, look at the terms work, energy, and heat, since they must be placed in consistent units before we can equate them.

Work is the ability of the flow of exert a force through some distance. The units of work in the engineering system are foot-pounds. Work can also be transmitted as torque on a moving shaft.

Energy is the *capacity* of the flow to do work. This capacity can be stored as kinetic, potential or internal energy. The latter is the kinetic energy contribution of random molecule motion. Kinetic and potential energy are normally measured in foot-pounds, while internal energy is measured in BTUs. By international agreement, the BTU is 778 foot-pounds. The conversion from ft-lbs to BTUs is done by applying "J" Joule's Constant = 778 foot-pounds/BTU.

Heat is a transient form of energy defined as the *energy in transfer* from one body to another by virtue of a temperature difference existing between the bodies. We normally measure heat in "BTUs of heat transferred".

Referring to Figure 10.3, the "General Energy Equation" begins as

$$Q - W_{1-2} = E_2 - E_1$$

The energy (i.e., the right-hand side) is comprised of five types of energy

1. **Potential Energy**. If the fluid leaves the system at some elevation different that the inlet, then there has been a change in *.PE*.

$$\Delta PE = W[Z_2 - Z_1]$$

2. **Kinetic Energy**. Between the inlet and the exit there may be a change in fluid velocity and hence a change in *KE*.

$$\Delta KE = \frac{1}{2}\frac{W}{g}\left[V_2^2 - V_1^2\right]$$

3. **Internal Energy**. The kinetic energy term about accounts for changes in the *average* velocities. In addition to this, a fluid molecule also has a random motion. Consider the schematic of a particle's flow as part of general fluid motion (Figure 10.4).

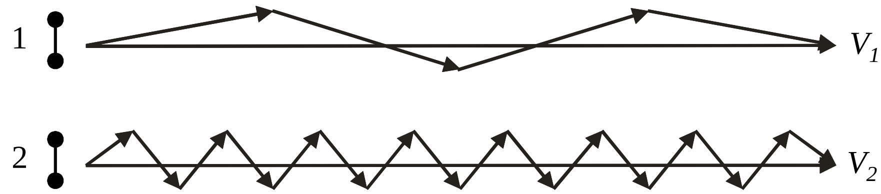


Figure 10.4 Low and High Random

In both of the examples above, the particle has the same average motion so, $V_1 = V_2$, from which we say the *kinetic* energies are the same. Particle number 2 however, has more random motion. This random motion is *solely a function of the absolute temperature*. The term which account for this energy of random motion is called internal energy (U).

The change in internal energy between entrance and exit is $\Delta E_{INT} = W(u_2 - u_1)$

Where lower case u denotes specific entry reduce $u = \frac{U}{W}$

The units of u are *BTU/lb* of fluid.

4 and 5. **Expansion and Flow Work**. The last two terms included in the evaluation of $E_2 - E_1$ are, the two "work" terms.

$$\text{Expansion Work} = w\int_1^2 Pdv$$

First consider the work done expanding or contracting a blob of matter as it goes from the entrance to the exit of a system. Picture the flow expanding against a piston, Figure 10.5.

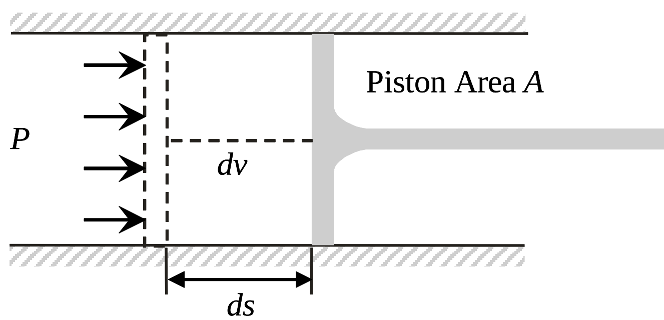


Figure 10.5 Flow Expanding Against a Piston

The force on the piston is $F = PA$

From basic mechanics, work is force applied through a distance, $dW=Fds=Pads$

Since $Ads=dv$

Therefore $dW=Pdv$

$$\text{Expansion Work} = w\int Pdv \qquad (10.11)$$

If dv is positive, the flow is expanding and the fluid is doing work on its environment. If dv is negative the flow is being compressed, and the environment is doing work on the fluid. Flow work is the work done in moving the chunk of matter from the entrance to the exit. If dp is positive, the pressure gradient is adverse and the flow is being forced "uphill" into a region of higher pressure, if dp is negative, the pressure gradient is favorable, and the flow is being "sucked" through the system.

$$\text{Flow Work } = \int_1^2 vdP \quad (10.12)$$

The expansion work and flow work done by or on the flow depends on the process involved. Incompressible and isentropic processes are of interest in evaluating expansion work.

a. For **incompressible processes** ($dv = 0$)

$$\text{Expansion work} = \int_1^2 Pdv = 0$$

$$\text{Flow work} = \int_1^2 vdP = v(P_2 - P_1)$$

b. For **isentropic processes** ($pv^\gamma = constant$)

$$\text{Expansion work} = \int_1^2 Pdv = \frac{P_2 v_2 - P_1 v_1}{1-\gamma}$$

$$\text{Flow work} = \int_1^2 vdP = \frac{\gamma}{\gamma - 1}\left[P_2 v_2 - P_1 v_1\right] \quad (10.7)$$

Again the conservation of energy could be written as

$$Q_{1-2} - W_{1-2} = E_2 - E_1$$

Expanding the right side with the 5 terms for the energy level, gives

$$Q_{1-2} - W_{1-2} = W(U_2 - U_1) + W\int_1^2 Pdv + W\int_1^2 vdP + \frac{W}{2g}\left(V_2^2 - V_1^2\right) + W(Z_2 - Z_1)$$

In normal engineering practice, some of the terms in this equation are measured in BTUs, some in foot-ponds. Applying the factor J = 778 ft-lb/BTU, gives an equation which is consistent in units and still assumes the normal units for each term. Dividing by the weight also, the equation has units of BTU/lb. And is known as **The General Energy Equation**.

$$q_{1-2} - w_{1-2} = u_2 - u_1 + \int_1^2 \frac{Pdv}{J} + \int_1^2 \frac{vdP}{J} + \frac{V_2^2 - V_1^2}{2gJ} + \frac{Z_2 - Z_1}{J} \quad (10.13)$$

The notation here is important. Capital letters refer to total quantities. Lower case letters refer to quantities per unit weight. Q and W are heat flux and work per unit mass, and U is specific internal energy, BTU/lb.

9.7.2 Enthalpy

The terms u, $\int \frac{Pdv}{J}$, and $\int \frac{vdP}{J}$ occur together frequently

It is convenient to lump these terms (internal energy, flow work and expansion work) together into one new term, enthalpy (h).

$$h \equiv u + \frac{Pv}{J} \quad (10.14)$$

In differential terms this becomes

$$dh = du + \frac{Pdv}{J} + \frac{vdP}{J} \tag{10.15}$$

In spite of the apparent complexity, this is actually a step in the direction toward simplifying the general energy equation.

Recall equation (10.1) for a perfect gas $Pv = RT$ (10.1)

Substituting this into Equation 10.14 gives $h = u + \frac{RT}{J}$

Since internal random energy *u* is a function only of temperature, then *h* can likewise be determined knowing only the fluid temperature. Integrating Equation 10.15 and substituting back into 10.13 gives another version of The General Energy Equation.

$$q_{1-2} - w_{1-2} = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2gJ} + \frac{Z_2 - Z_1}{J} \tag{10.16}$$

This equation can be rearranged to separate states 1 and 2:

$$q_{1-2} - w_{1-2} = \left(h + \frac{V^2}{2gJ} + \frac{Z}{J} \right)_2 - \left(h + \frac{V^2}{2gJ} + \frac{Z}{J} \right)_1$$

Where the time terms in parenthesis now specify the total energy level (enthalpy, kinetic, and potential) at states 1 and 2.

9.7.3 Special Cases of the General Energy Equation

If there is no heat transfer during some process it is called an adiabatic process. In this case $\Delta q = 0$, so the *q* term may be neglected in the general energy equation. Flow in intakes, compressors and exhausts are good approximations to adiabatic flow. Note that adiabatic flow is not necessarily constant temperature flow.

If the fluid flows at constant temperature, it is called *isothermal* flow, $h_2 - h_1 = 0$ from the definition of enthalpy.

9.7.4 The Heat Equation

Subtracting the momentum Equation (10.5) from the general energy Equation (10.13) gives:

$$q_{1-2} = u_2 - u_1 + \int_1^2 \frac{Pdv}{J} \tag{10.17}$$

This is called "The Heat Equation" and applies to any flow process or non-flow process. Since q_{2-1} and u_2-u_1 are dependent only on temperature, it follows that temperature also specifies the value of the remaining term $\left(\int_1^2 \frac{Pdv}{J} \right)$. Armed with this information we are ready to discuss specific heat in some detail.

9.7.5 Specific Heat

Specific heat constant volume process: The combustion of fuel in the cylinder of a reciprocating engine is an example of a heat addition process in which there is no change in volume. Apply Equation 10.17.

For a constant volume, $dv = 0$ giving $[q_{1-2}]V = \text{constant} = u_2 - u$

The amount of heat added is equal to the change in internal energy. We previously defined the specific heat at constant volume (C_v) as the amount of heat added per pound of fluid required to raise its temperature one degree Fahrenheit during a constant volume process.

Therefore: $$q_{1-2} = u_2 - u_1 = C_v(T_2 - T_1) \tag{10.18}$$

$$\boxed{\Delta u = C_v \Delta T}$$

Specific heat in a constant pressure process: The combustion of fuel in a jet engine burner is an example of a heat addition process which occurs at nearly constant pressure.

Start with the heat equation

$$q_{1-2} = u_2 - u_1 + \int_1^2 \frac{Pdv}{J} \tag{10.17}$$

Apply the definition of enthalpy: $$dh = du + \frac{pdV}{J} + \frac{vdP}{J} \tag{10.15}$$

Integrate and subtract to get: $$q_{1-2} = h_2 - h_1 - \int_1^2 \frac{vdP}{J}$$

Since $dp = 0$ in a constant pressure process $[q_{1-2}]_{p = \text{constant}} = h_2 - h_1$

By definition, the specific heat at constant pressure (C_p) is the amount of heat flux required to raise the temperature of one pound of fluid by one degree Fahrenheit in a constant pressure process

$$q_{1-2} = C_p(T_2 - T_1)$$

or

$$q_{1-2} = h_2 - h_1 = C_p(T_2 - T_1)$$

Therefore $$\boxed{\Delta h = C_p \Delta T} \tag{10.19}$$

Using these relationships and recalling the definition of the specific heat ratio, $\gamma \equiv C_p / C_v$, several useful relationships can be written:

Starting with Equation 10.14 $$h = u + \frac{Pv}{J}$$

Apply Equations 10.18 and 10.19 to get $$C_p T = C_v T + \frac{RT}{J}$$

Divide through by T $$C_p = C_v + \frac{R}{J} \text{ or } C_p - C_v = \frac{R}{J} \tag{10.27}$$

Divide through by C_v $$\frac{C_p}{C_v} - \frac{C_v}{C_v} = \frac{R}{JC_v}$$

Recall that $\gamma \equiv \frac{C_p}{C_v}$ to get $$\gamma - 1 = \frac{R}{JC_v}$$

Finally, $$C_v = \frac{R}{J}\frac{1}{(\gamma - 1)} \text{ and } C_p = \frac{R}{J}\frac{\gamma}{\gamma - 1} \qquad (10.20)$$

We have shown that the enthalpy change for a constant pressure process is given by $\Delta h = C_p \Delta T$. The internal energy change for a constant volume process is $\Delta u = C_v \Delta T$. In fact, these calculations are good for any process. Proof of this is straightforward and makes use of the fact that *h* and *u* are defined only by the temperature of the fluid. The process by which the fluid arrived at the temperature is therefore not material in determining *h* and *u*.

Degrees of Freedom

For air $C_p = 0.241$

$C_v = 0.173$

From the kinetic theory $$\gamma = \frac{f+2}{f}$$

where *f* is the number of degrees of freedom of the molecule.

Any molecule is free to translate along 3 axis; hence there are 3 degrees of freedom in translation for all molecules (Figure 10.6).

A diatomic molecule (like nitrogen, *N* and oxygen, 0_2) is also free to absorb momentum by rotating about 2 of its 3 axis.

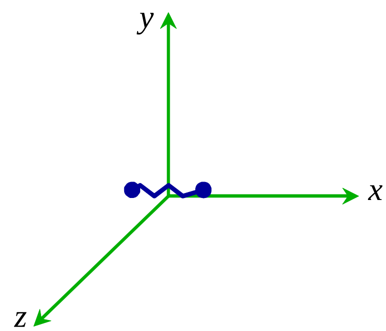


Figure 10.6 Degree of Freedom of a Diatomic Molecule

In this picture the molecule can rotate about the *y* and *z* axis and store momentum, but it cannot rotate about the *x* axis since the moment of inertia, I_x is essentially zero. Hence, no stored angular momentum. For air at standard temperature, then, the molecules have five degrees of freedom. The equation $\gamma = \frac{f+2}{f}$, would predict a γ of $\frac{7}{2}$ or 1.4 which is the correct value for air. At extreme temperatures, the excitation of more degrees of freedom will cause the specific heat ratio to change. This topic is covered in the course on hypersonic aerodynamics.

Example Problem 1: One pound of gas which has a molecular weight of 33 is heated from 500° R to 1500° R at constant pressure by the absorption of 250 BTU of heat. Calculate its value of R, C_p, C_v, γ, Δu and Δh.

$$R = \frac{1544}{molecular\ \ wt} = \frac{1544}{33} = 51.5\frac{ft\,lb}{lb\ {}^o\mathrm{R}}$$

$$C_p = \frac{q}{\Delta T} = \frac{250}{1000} = .25\frac{BTU}{lb\ {}^oR}$$

$$C_v = C_p - \frac{R}{J} = .25 - \frac{51.5}{778} = 0.1838\frac{BTU}{lb\ {}^oR}$$

$$\gamma = \frac{C_p}{C_v} = \frac{.25}{.1838} = 1.36$$

$$\Delta u = C_v \Delta T = .1838 \times 1000 = 183.8\frac{BTU}{lb}$$

$$\Delta h = C_p \Delta T = .25 \times 1000 = 250\frac{BTU}{lb}$$

Example Problem 2: A turboprop engine is operated in standard sea level air at Mach 0.5 and a shaft output of 1000 HP (= 550,000 $\frac{\text{ft lb}}{\text{sec}}$). If the exit jet velocity and temperature are 1200 ft/sec and 1040°F, respectively. Assume that the weight of the fuel added is negligible compared to the weight of the air.

a. Calculate the amount of heat added to each pound of air taken aboard at the rate of 64.4 lb/sec.
b. Calculate the jet thrust developed.

Start with the energy equation $q_{1-2} - w_{1-2} = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2gJ} + \frac{Z_2 - Z_1}{J}$ (10.16)

Where the last term can be neglected. Solving for w (in units of BTU/lb)

$$w = \frac{W}{T}\frac{1}{J} = \frac{1000\,HP}{G} = 1000\,HP \times 550\frac{ft\,lb}{\sec HP} \times \frac{1}{778}\frac{BTU}{ft\,lb} \times \frac{1}{64.4}\frac{\sec}{lb} = 11.0\frac{BTU}{lb\ Air}$$

$$h_2 - h_1 = C_p\left(T_2 - T_1\right) = 0.24(1500 - 519) = 235 BTU/lb$$

$$V_1 = M\sqrt{1.4 \times 53.3 \times 32.2 \times 519} = 560\,ft/\sec$$

a) $$q_{1-2} = w_{1-2} + h_2 - h_1 + \frac{V_2^2 - V_1^2}{2gJ} = 11.0 + 25 + \frac{1200^2 - 560^2}{2 \times 778 \times 32.2} = 268.4\frac{BTU}{lb}$$

b) $$F = \frac{G}{g}\left(V_2 - V_1\right) = \frac{64.4}{32.2}(1200 - 560) = 1280 lb$$

9.8 Second Law of Thermodynamics

In essence, the 2nd law limits the amount of heat which can be converted to work. The first law gives a statement of the energy balance which must hold true if heat flux is converted to work or vice versa. The fact that a given hypothetical process would satisfy the first law is no guarantee, however, that such a process can be carried out. All natural processes have a preferred direction; rivers run downhill, people grow older, if you put pig into the meat grinder you get out sausage. The reverse of these processes is never observed. When an automobile is stopped by a friction brake the brake gets hot; the internal energy of the brake increases as it absorb the kinetic energy of the vehicle satisfying the first law. The first law, however, would be equally well satisfied if the brake suddenly cooled, giving up its internal energy as kinetic energy, causing the car to accelerate. This latter process never happens. The braking action of a car is an *irreversible* process.

A process is reversible if, after it has been carried out, it is possible by any means to restore both the system and its entire surrounding to exactly the same states they were in before the process. All real processes are irreversible, but some come close to being reversible that they can be accurately treated as reversible. A free pendulum, for example, very nearly trades kinetic for potential energy in a reversible fashion. Experience shows that any of five circumstances during a process will render it irreversible.

1 Heat transfer through a finite temperature difference.

2 Lack of pressure equilibrium between a system and its confining walls.

3 Free expansion of a system to a larger volume in the absence of work done by the system.

4 Solid or fluid friction (or electrical resistance).

5 Transfer of paddle-wheel work to a system.

Mechanical energy can be converted to thermal energy (heat flux or a rise in internal energy). Irreversibility occurs when we cannot completely convert thermal energy back into mechanical energy. Some of the energy has become unavailable. Note that energy has not been destroyed. It still exists, but it has eluded our grasp. The second law of thermodynamics is a general statement about the increase of unavailability of energy (entropy) which may occur during any process. Although difficult to pin down precisely, entropy may also be thought of as the degree of irreversibility of a process. A more rigorous entropy definition is possible after we establish a few more terms.

9.8.1 Cycle Efficiency

Before developing the entropy concept, some definitions must first be stated. A cycle is a series of processes, which may be repeated, in a given order, the working fluid passing through various state changes and returning periodically to its initial state.

A reversible cycle is one made up of reversible processes. Figure 10.7 shows the schematic for such a cycle.

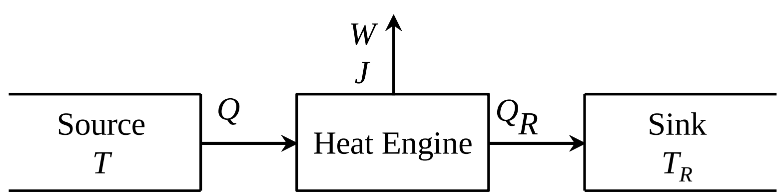


Figure 10.7 Heat Cycle Schematic

A heat engine is a fluid cycle, the object of which is to work from heat. Three elements are always necessary in a heat engine:

1. The reception of energy as heat from a high-temperature source.
2. The delivery of some of this enegy as work.
3. The rejection of the remainder of the energy as heat to a lower temperature receiver.

The cycle efficiency is the ratio of the energy output to input (or the ratio work delivered to the heat supplied).

Let Q = heat supplied from the source

Q_R = heat rejected to the sink.

W/J = work delivered by the cycle.

Then for any cycle, reversible or unreversible

$$Q = Q_R + W / J$$

Efficiency
$$\eta = \frac{W / J}{Q} = \frac{Q - Q_R}{Q} = 1 - \frac{Q_R}{Q} \qquad (10.21)$$

9.8.2 The Carnot Principle

In 1824 Sadi Carnot established the necessary conditions for the maximum efficiency of a heat engine. His principle can best be stated by the following 3 propositions:

1 No cycle which continuously delivers work by accepting energy at a high temperature and rejecting the residue at a lower temperature can be more efficient than a reversible cycle operating between a source and a sink at these temperatures.

2 The efficiency of a reversible cycle depends only on the temperatures of the source and the receiver.

3 The efficiency of all reversible cycles working between the same temperatures is the same regardless of differences in the cycles or the working fluids.

Kelvin used this second principle to establish the concept of absolute temperature. Using a perfect gas in a reversible cycle, the absolute temperature scale was defined by

$$\frac{T_R}{T} = \frac{Q_R}{Q}$$

Where T_R = absolute temperature of the sink

T = absolute temperature of the source

Since the efficiency of any cycle is $1 - \frac{Q_R}{Q}$, then the efficiency of any <u>reversible</u> cycle is $1 - \frac{T_R}{T}$

Carnot went on to define a specific reversible cycle of some interest as the ideal cycle. The Carnot cycle is valid for any working fluid and is defined by four processes, as follows:

1. A reversible constant temperature expansion in which some heat flux Q is received from a source at constant temperature T.
2. A reversible adiabatic expansion in which the fluid passes from the source temperature T to the sink temperature T_R.
3. A reversible isothermal compression in which some heat flux Q_R is rejected to a sink at temperature T_R.
4. A reversible adiabatic compression in which the fluid is returned from the sink temperature T_R, to the source temperature T, its original state.

During steps 1 and 2 the fluid does work on its environment. During steps 3 and 4 the environment does work on the fluid.

Since the ideal Carnot cycle is reversible, its efficiency is calculated as $1-\frac{T_R}{T}$

9.8.3 Entropy

The rejected heat flux Q_R at temperature T_R represents the energy unavailable to the cycle. The only way to convert all of the heat energy into an available (mechanical) form is to reject zero heat flux, i.e., have a heat sink at zero degrees absolute. The corresponds to 100% Carnot efficiency. As stated before, for a reversible cycle

$$\frac{Q_R}{Q}=\frac{T_R}{T} \quad \text{or} \quad \left.\frac{Q_R}{T_R}\right|_{REV}=\frac{Q}{T}$$

According to the Carnot principle a reversible process is the most efficient possible process operating between T and T_R. It follows, therefore that for an irreversible process more energy will be unavailable. Hence:

$$\left.\frac{Q_R}{T_R}\right|_{IRREV}>\frac{Q}{T}$$

Now relax the constant temperature restriction and allow it to vary from T_1 toT_2. The above inequality becomes the Inequality of Clasius:

$$\left.\frac{Q_R}{T_R}\right|_{IRREV}>\int_1^2\frac{dQ}{T}$$

similarly,

$$\left.\frac{Q_R}{T_R}\right|_{REV}=\int_1^2\frac{dQ}{T}$$

Because of its use in evaluating the availability of energy for a reversible process, Clasius named $\int\frac{dQ}{T}$ the entropy.

$$\left.s_2-s_1\right|_{REV}=\int_1^2\frac{dQ}{T} \quad \text{or} \quad Tds=dq \tag{10.22}$$

This equation does <u>not</u> apply to irreversible processes. It is, however, a simple matter to visualize some reversible path between any states 1 and 2, and then calculate the entropy change. The entropy is a property of a fluid and that the change is entropy between any two states is independent of the path. The next step is to expand the integral into measurable quantities.

Rewriting the heat Equation (10.17) in differential form

$$dq=du+\frac{Pdv}{J} \tag{10.23}$$

This is also commonly described as a statement of the first law for mechanically reversible processes. Recall the definition of specific heats $C_v \equiv du/dT$ or du = $C_v\ dT$. Substituting this and Equation (10.22) into the above equation gives:

$$Tds=dp=C_v dT+\frac{P}{J}dv$$

Dividing through by T

$$ds=C_v\frac{dT}{T}+\frac{1}{J}\frac{P}{T}dv$$

Rearrange the perfect gas law (10.1) to get $\frac{P}{T} = \frac{R}{v}$

Substituting into the above equation gives

$$ds = C_v \frac{dT}{T} + \frac{R}{J}\frac{dv}{v}$$

Integrating gives a way to determine the entropy changes BTU/lb oR between two states

$$\boxed{s_2 - s_1 = C_v \ln\frac{T_2}{T_1} + \frac{R}{J}\ln\frac{v_2}{v_1}} \qquad (10.24)$$

Note that the entropy change can be defined knowing only the volume and temperature of the initial and final states. We can also develop another relationship for s_2-s_1. Again starting with the differential heat Equation

$$dq = du + \frac{Pdv}{J}$$

Where Equation (10.15) $dh = du + \frac{Pdv}{J} + \frac{vdP}{J}$

Can be arranged as $du + \frac{Pdv}{J} = dh - \frac{vdP}{J}$

Substituting this into Equation (10.23) gives

$$dq = dh - \frac{vdP}{J}$$

Dividing this through by *T* gives $\frac{dq}{T} = \frac{dh}{T} - \frac{v}{T}\frac{dP}{J}$

Substitute Equation (10.19) into the first term on the RHS $\left(\Delta h = C_p dT\right)$

$$\frac{dq}{T} = C_p \frac{dT}{T} - \frac{R}{P}\frac{dP}{J}$$

From the perfect gas law we know $\frac{\gamma}{T} = \frac{R}{P}$

According to Equation (10.22), integrating the RHS gives the change in entropy

$$S_2 - S_1\big|_{REV} = \int_1^2 \frac{dq}{T} \qquad (10.22)$$

$$\boxed{s_2 - s_1 = C_p \ln\frac{T_2}{T_1} - \frac{R}{J}\ln\frac{P_2}{P_1}} \qquad (10.25)$$

This description of the entropy change requires knowledge of the pressure and temperature of the initial and final states

9.8.4 Poisson's Relations

If we set s_2-s_1=0 we can now develop the isentropic relationships.

$$C_v \ln \frac{T_2}{T_1} + \frac{R}{J} \ln \frac{v_2}{v_1} = 0$$

Substituting the perfect gas law relation $\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{v_2}{v_1}$

gives $$C_v \ln \frac{P_2}{P_1}\frac{v_2}{v_1} = -\frac{R}{J} \ln \frac{v_2}{v_1}$$

$$C_v \left[\ln \frac{P_2}{P_1} + \ln \frac{v_2}{v_1} \right] = -\frac{R}{J} \ln \frac{v_2}{v_1}$$

Substitute Equation (10.20) $C_v = \frac{R}{J}\frac{1}{\gamma - 1}$

to get $$\left(\frac{R}{J}\right)\left(\frac{1}{\gamma-1}\right)\left[\ln \frac{P_2}{P_1} + \ln \frac{v_2}{v_1} = -\frac{R}{J} \ln \frac{v_2}{v_1} \right]$$

Divide through by $\frac{R}{J}$ $$\left(\frac{1}{\gamma-1}\right)\left[\ln \frac{P_2}{P_1} + \ln \frac{v_2}{v_1} \right] + \ln \frac{v_2}{v_1} = 0$$

Multiply through by $$(\gamma - 1)\ln \frac{P_2}{P_1} + \ln \frac{v_2}{v_1} + (\gamma - 1)\ln \frac{v_2}{v_1} = 0$$

Combine terms $$\ln \left[\left(\frac{P_2}{P_1}\right)\left(\frac{v_2}{v_1}\right)^{\gamma} \right] = 0$$

Finally Equation (10.25) can be interpreted as $P_2 v_2^{\gamma} = P_1 v_1^{\gamma} = \text{Constant}$ (10.26)

This is the general expression for an isentropic process. Combining Equation (10.26) and the perfect gas law (10.1), following can also be derived:

$$\left.\begin{aligned} \frac{P_2}{P_1} &= \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \\ \frac{v_2}{v_1} &= \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} \\ \frac{T_2}{T_1} &= \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} \end{aligned}\right\} ISENTROPIC$$

9.8.5 Applications of The Second Law

Example: Calculate the entropy change for the combustion process in a jet engine (at constant pressure) if the air enters at 350°F leaves at 2000°F (C_p=.24)

$$s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - \frac{R}{J} \ln \frac{P_2}{P_1} \qquad (10.25)$$

Since the second term is zero

$$s_2 - s_1 = .24 \ln \frac{2460}{810} = .24 \ln 3.16 = .276 BTU / lb°R$$

For a constant volume process, the alternate form of the equation is obviously easier to use.

$$s_2 - s_1 \big|_{v=constant} = C_v \ln \frac{T_2}{T_1} + \frac{R}{J} \ln \frac{v_2}{v_1} \quad (10.24)$$

A plot of temperature versus the entropy is a useful tool for analyzing an engine cycle. Constant pressure lines appear as shown in Figure (10.8). Enthalpy can be substituted for temperature on the ordinate since the two differ only by a constant, as seen from $h=C_pT$.

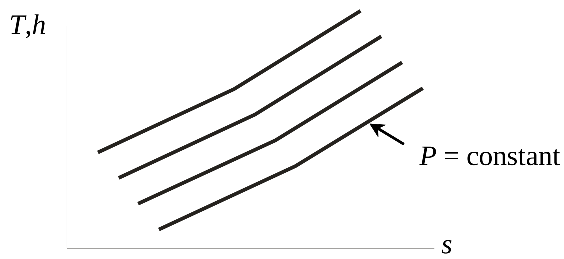


Figure 10.8 *T-s,h-s* Diagram

Figure 10.9 shows the process of change temperature and entropy as air moves through a turbine engine.

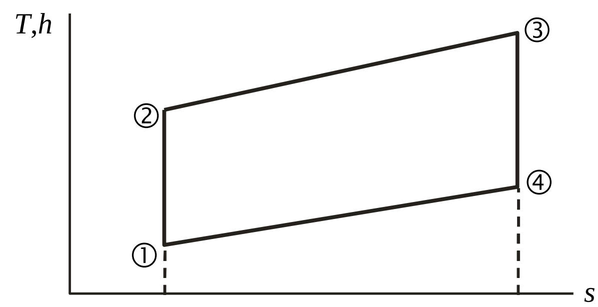


Figure 10.9 Turbine Engine Cycle

1. Air enters an engine inlet at some temperature and reference level of entropy.
2. The air is adabatically compressed by the inlet and the compressor to some higher temperature, T_2, but at approximately constant entropy.
3. From 2 to 3 heat is added in the combustion chamber. T and s increase, but pressure remains approximately constant.
4. The flow expands adabatically expanded through the turbine and exhaust nozzle. The temperature decreases, but the entropy remains approximately constant.

Since the pressure at both 4 and 1 is ambient we can arbitrarily close the curve since we can move air to or from the surroundings as we desire.

The close area within the curve is $\int Tds$. The definition of the entropy states.

$$ds = Tds$$

Therefore the area within the curve is that heat flux q, available as useful work. The area below the closed curve, and within the dotted lines is the heat flux rejected to the atmosphere, or unavailable energy.

9.9 References

10.1 Onizuka, Ellison "Thermodynamics Review" USAF Test Pilot School, Edwards AFB, CA, 1977.

10.2 Schob, W.J., Schwellkhard W.G., "Aerodynamics Theory" FTC-TIH-64-2007, Edwards AFB, CA, 1964

Volume 1 – Math & Physics for Flight Testers

Chapter 10

Spreadsheets in Flight Test

Table of Contents

10.1 Introduction

Large, thousand-plus-page books are sold showing how to use Microsoft Excel for various tasks. While obviously not as comprehensive, these notes focus on the use of spreadsheets in flight test, both for test planning and for data reduction. A series of hands-on exercises in the classroom will be used to reinforce the elements covered during the lecture and in these notes. Hopefully these notes plus example spreadsheets will give the test pilot and flight test engineer the tools needed to facilitate flight test planning and post-flight data analysis during the larger flight test course.

10.2 The Basics

10.2.1 Cell References and Equations

Cells in a spreadsheet can be referred to several different ways, but the most common way is by column letter + row number (e.g. "Al" for the top left cell in a spreadsheet). When entering data into a cell the data will be treated as a number or label unless the entry is preceded by an equals symbol ("="). If the entry is preceded by an equals symbol then the entry is interpreted as an equation and the cell value will be set to a numerical value corresponding to the result of the equation. In the following example (see Figure 11.1), cell D3 is the active, or selected, cell as noted by the heavy border plus the "D3" showing in the Name Box just below the font type (Arial), plus the raised effect on the D column and the 3rd row. The contents of the active cell are displayed in the Formula Bar just above the column headers and to the right of the Name Box. For cell D3 it shows "=B3+C3" which means that the cell value of D3 will be set to the sum of cells B3 and C3. Thus, in the example, cell D3's value is set to 3, the sum of 1 and 2.

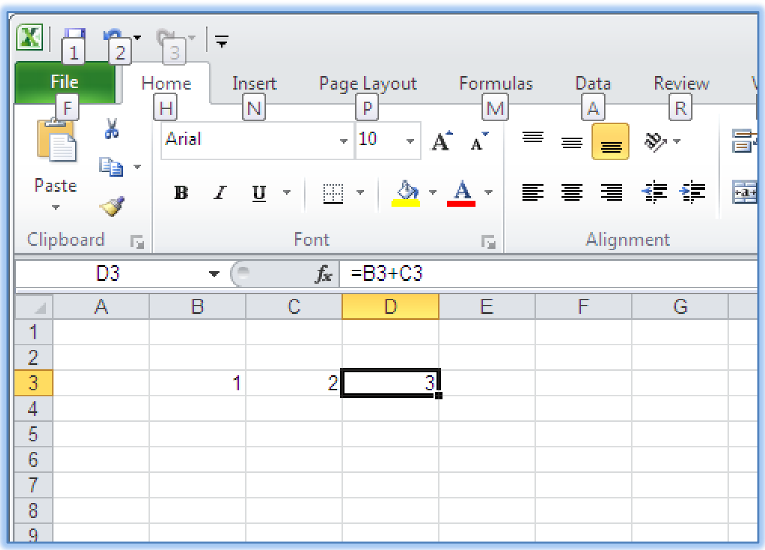


Figure 11.1 Example Equation

In Excel equations, all of the normally expected operators work:

a. + for add

b. - for subtract

c. / for divide

d. * for multiply

The ∧ symbol is used for raising to a power, so "=B3^2" in a cell would produce the square of cell B3's value. This can also be used for square roots by doing "=B3^0.5". An alternative way to take a square root

is to use one of the many built in, standard Excel functions, namely SQRT ("=SQRT(B3)"). There will be more on functions later in these notes.
Ranges of cells can also be referred to as a group. The use of (B2:B5) in an equation would refer to the range of four cells vertically from B2 extending down to B5. The use of cell ranges inside an equation works well with built in functions such as SUM. This is illustrated in Figure 11.2 below. Cell B6 in the example has an equation that uses the Excel SUM() function along with a reference to the cell range B2:B5.

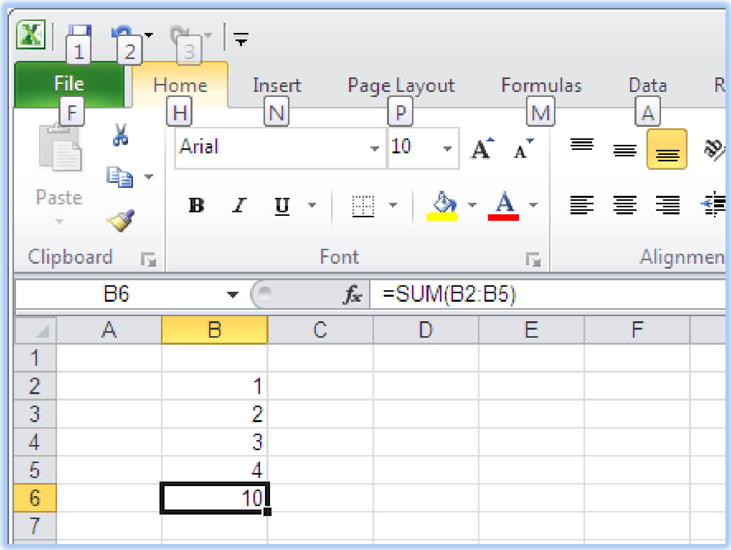


Figure 11.2 Use of Cell Ranges

A shortcut to adding up the contents of a column (or row) of numbers is to select the cell just below (or to the right) of the group of cells to be added. Then click on the Greek letter Σ on the toolbar. The Σ is on the right side of the Home menu tab. Selecting the Σ with B6 as the active cell will result in the equation you see in Figure 11.2 above, being offered by the program, requiring you simply to hit the <Enter> key to accept the offered equation.

10.2.2 Copying Equations

Duplicating an equation for a group of rows or columns can greatly simplify repeated calculations. Referring to Figure 11.3, the contents of cell D3 is again "=B3+C3", as it was in Figure 11.1. Selecting D3, commanding a copy, and then pasting the results in cells D4, D5 and D6 results in these cells being the sum of the two cells to their respective left. As shown in Figure 11.3, cell D6's contents are "=B6+C6." Thus the copy/paste action defaults to a relative addressing paste. Cell DX contains a relationship that says "add the cell two to my left (BX) to the cell one to my left (CX) and put the total in this cell (DX)".
There are multiple ways to copy cells in Excel. One way is to use the toolbar icons. With the desired cell selected, putting the mouse over the copy icon (the two pages symbol, just below the Tools menu item in Figure 11.3) and hitting the left mouse button will command a copy. Selecting a different cell and hitting the paste icon (to the right of the copy icon) will effect a copy paste action. Selecting Edit on the menu followed by Copy and then subsequently doing Edit/Paste will do the same. Both of these methods are easy for the new user but are time consuming for the "power" user. Most proficient users prefer the keyboard shortcuts <Ctrl+C> for copy and <Ctrl+V> for paste. Possibly the best of all, though, is to use the "handle" on an active cell to copy and paste in one movement. The small open square at the bottom right of an active cell is a handle that if you left-click on and drag, you will automatically copy and paste in one motion.

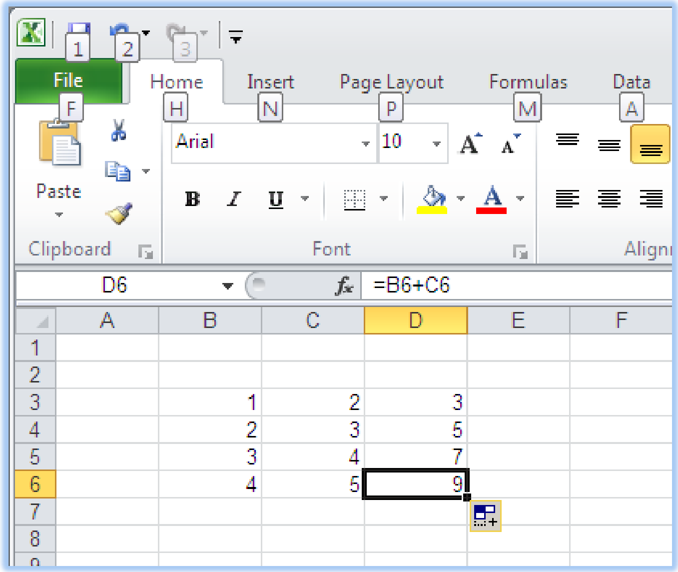


Figure 11.3 Copying Equations

10.2.3 Formatting

Formatting is not just for making a spreadsheet attractive. The use of standards can make a spreadsheet easier to use and less prone to input errors. The standard cell fill colors used at NTPS are as follows:

a. gray - default (empty cell or intermediate calculations)
b. light green - labels and titles
c. light yellow - calculated values, results
d. white - data entry areas

The weight and balance spreadsheet shown next illustrates the use of these colors.

Merlin N75MX Center of Gravity

	Weight (lbs)	Arm (in)	Moment
Basic A/C	7,958	162.7	1,294,606
Pilot	180	111.0	19,980
Copilot	0	111.0	0
Seat 1	165	149.0	24,585
Seat 2	0	165.8	0
Seat 3	0	182.3	0
Seat 4	0	198.8	0
Seat 5	0	222.7	0
Seat 6	182	242.0	44,044
Seat 7	0	261.4	0
Fuel	1,800	183.0	329,400
Fwd Baggage	150	42.0	6,300
Aft Baggage	0	320.0	0
Ballast	0	0.0	0
TOTAL	10,435	164.7	1,718,915
% MAC		23.2%	

Figure 11.4 Use of Standard Colors

Obviously the use of these standard colors is not mandatory but it does directly lead the user to the white areas as input slots. When new data is entered into any white area, the two yellow boxes give the immediate result of the new total weight and center of gravity.

To change the cell colors, border, alignment, etc., select the cell(s) and then use the "Format/Cells..." menu command. Alternatively, right-click on a cell or group of cells and use the popup menu, "Format Cells..." selection. Either way, you'll end up with the following dialog box (Figure 11.5) that will allow formatting of the cell(s). Note - to select all cells in a worksheet (to set the default to a gray background, for example) left-click on the square above the row numbers and to the left of the column letters. A shortcut to selecting all cells is <Ctrl A>.

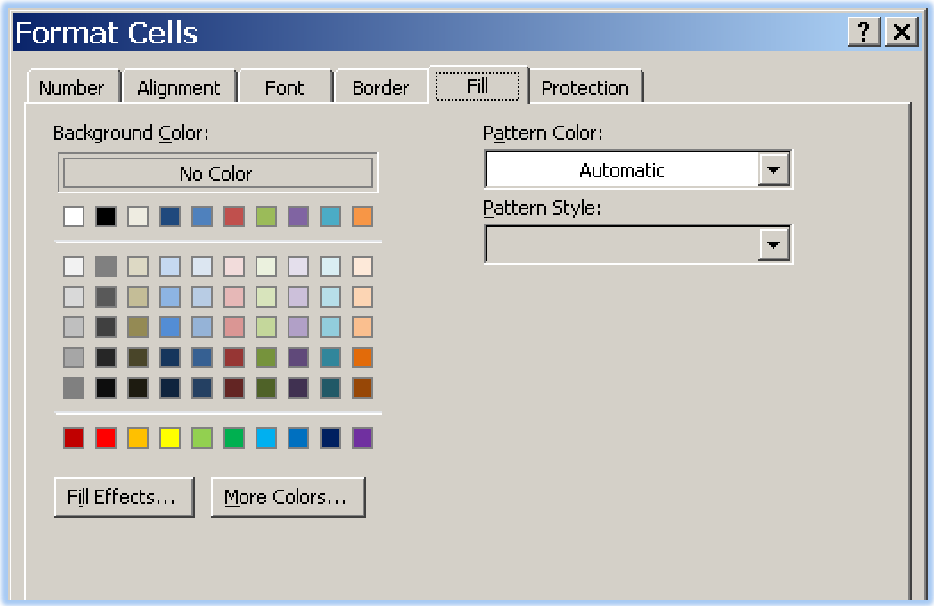


Figure 11.5 Cell Formatting Dialog Box

10.2.4 Cell Protection

While the above standard colors lead the user to input data into the proper areas, it does not prevent a user from typing a number into a cell that has an equation, thus destroying the relationships needed to give the proper answers. A way to prevent these types of errors is to use the cell protection features of Microsoft Excel. Note that "Protection" is one of the tabs in the formatting dialog box in Figure 11.5 above. If you check any cell's protection property on the using the above dialog box you will find that the default for protection is "Locked." But of course, the cell contents are not default locked otherwise you wouldn't be able to type anything in the cell. The reason is that for a cell to be locked, both the protection property must be set to locked and the worksheet must be protected at the same time.

Thus, going back to the weight and balance example shown in Figure 11.4, after the spreadsheet is complete, you want to protect everything except the white areas (user input errors). To do this, select the white areas - click on C6 and drag to C18, then holding down the <Ctrl> key, click on D18. The holding of the <Ctrl> key is necessary when selecting non-contiguous cell groups. After selecting all of the white cells, format the protection property to not locked. Then use File tab, Info submenu item to turn protection on for the desired worksheet or the entire workbook. After choosing protection you will be presented with another dialog box allowing you to select a password for later changes. If the concern is just avoiding inadvertent typing in the wrong cell, hit Okay without a password and then no password will be required if later you want to turn the protection off.

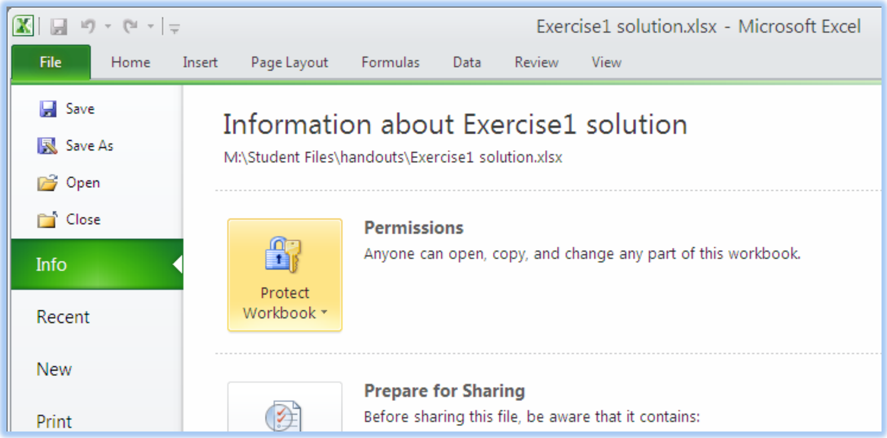


Figure 11.6 Invoking Cell Protection

10.2.5 Exercise 1

Build a spreadsheet that replicates the one in Figure 11.4. Use the same formatting scheme, including cell protection as described above. The individual component's moment is the individual component's weight multiplied by its respective arm. The total weight and total moments are the sums of their respective columns. The "total arm" is the total moment divided by the total weight. The %MAC is the (total arm - 147 in) divided by 76.45 inches. In the Merlin the leading edge of the mean aerodynamic chord (MAC) is 147 inches aft of the datum plane and the length of the MAC is 76.45 inches.

10.3 Graphing

In flight test, a good graph can communicate a large amount of data efficiently. What used to occupy an entire department in a flight test organization can now be done quickly and easily by the flight test engineer or pilot using the graphing capabilities of Excel. Figure 11.7 shows a completed graph that not only shows the test results but also shows their tendencies and then-relationship to the requirements.

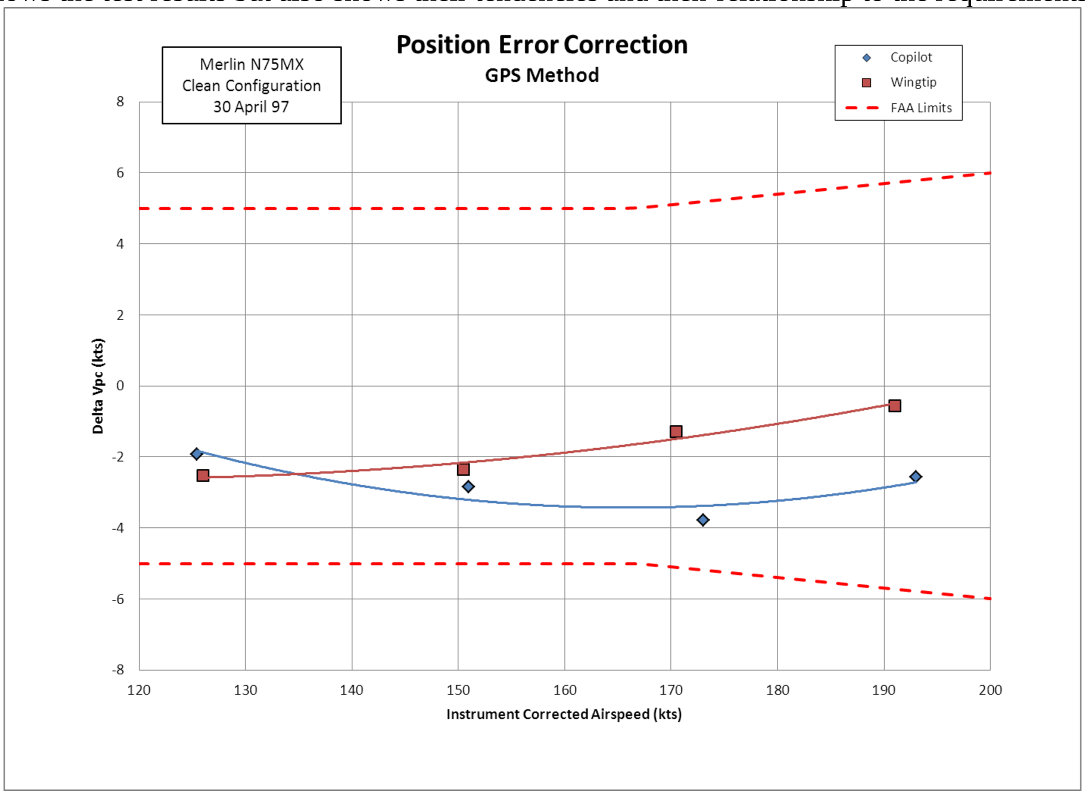
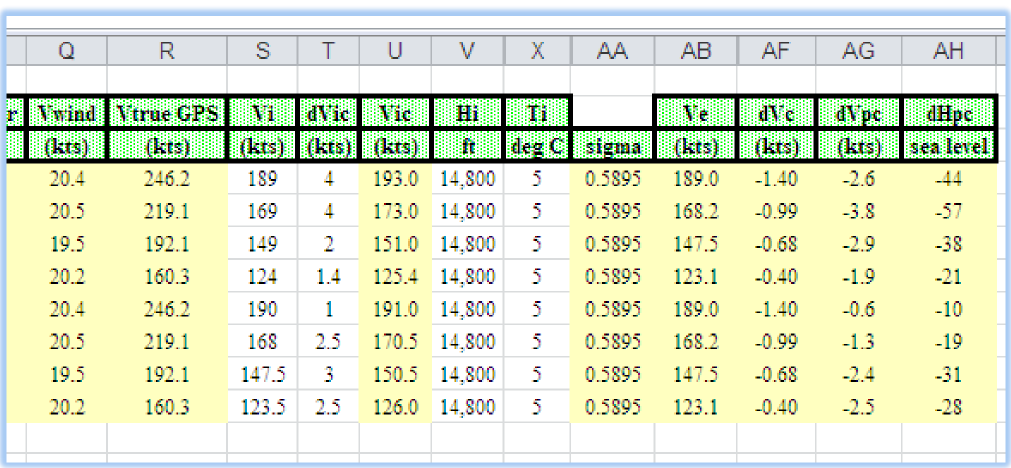


Figure 11.7 Example of Excel's Graphing Capabilities

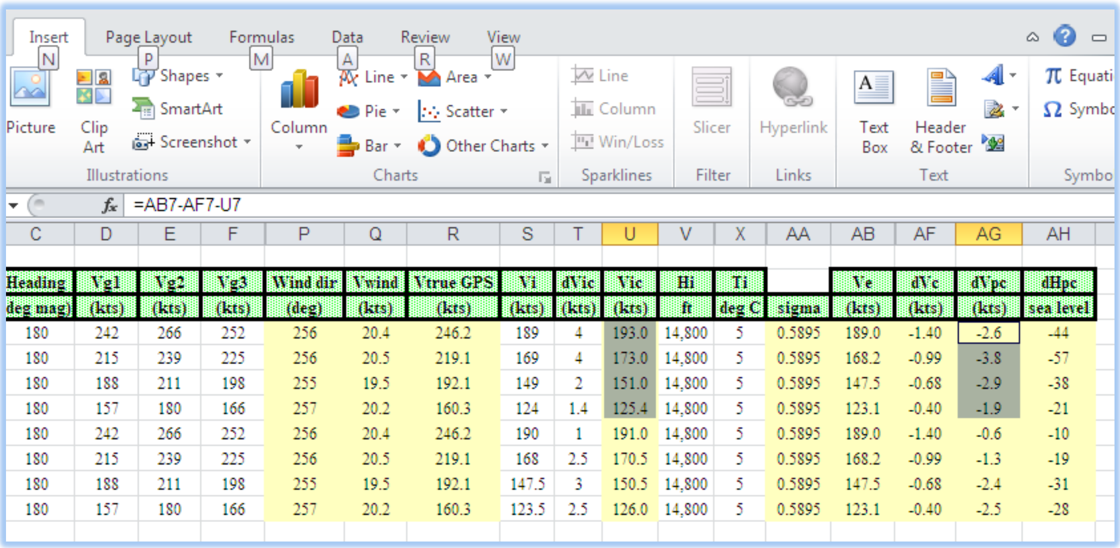


Q	R	S	T	U	V	X	AA	AB	AF	AG	AH
Vwind (kts)	Vtrue GPS (kts)	Vi (kts)	dVic (kts)	Vic (kts)	Hi ft	Ti deg C	sigma	Ve (kts)	dVc (kts)	dVpc (kts)	dHpc sea level
20.4	246.2	189	4	193.0	14,800	5	0.5895	189.0	-1.40	-2.6	-44
20.5	219.1	169	4	173.0	14,800	5	0.5895	168.2	-0.99	-3.8	-57
19.5	192.1	149	2	151.0	14,800	5	0.5895	147.5	-0.68	-2.9	-38
20.2	160.3	124	1.4	125.4	14,800	5	0.5895	123.1	-0.40	-1.9	-21
20.4	246.2	190	1	191.0	14,800	5	0.5895	189.0	-1.40	-0.6	-10
20.5	219.1	168	2.5	170.5	14,800	5	0.5895	168.2	-0.99	-1.3	-19
19.5	192.1	147.5	3	150.5	14,800	5	0.5895	147.5	-0.68	-2.4	-31
20.2	160.3	123.5	2.5	126.0	14,800	5	0.5895	123.1	-0.40	-2.5	-28

Figure 11.8 Data for Figure 11.7 Graph

10.3.1 Creating a Chart

First select the data to be graphed. In this example, make cell U7 the active cell by left-clicking on it with the mouse and then dragging down to cell U10. This will be the x-axis data for the first series. A series is a group of like data. We don't include U11 through U14 initially because the "Wingtip System" on the Merlin aircraft is a different pitot-static system and therefore should be graphed separately. Now we need to also select the y-axis data at the same time. To simultaneously select a non-contiguous range of cells, hold down the <Ctrl> key and drag the mouse pointer from AG5 to AG8. In the end you should have cells U7:U10 and cells AG7:AG10 both highlighted as shown in Figure 11.9.



Heading deg mag	Vg1 (kts)	Vg2 (kts)	Vg3 (kts)	Wind dir (deg)	Vwind (kts)	Vtrue GPS (kts)	Vi (kts)	dVic (kts)	Vic (kts)	Hi ft	Ti deg C	sigma	Ve (kts)	dVc (kts)	dVpc (kts)	dHpc sea level
180	242	266	252	256	20.4	246.2	189	4	193.0	14,800	5	0.5895	189.0	-1.40	-2.6	-44
180	215	239	225	256	20.5	219.1	169	4	173.0	14,800	5	0.5895	168.2	-0.99	-3.8	-57
180	188	211	198	255	19.5	192.1	149	2	151.0	14,800	5	0.5895	147.5	-0.68	-2.9	-38
180	157	180	166	257	20.2	160.3	124	1.4	125.4	14,800	5	0.5895	123.1	-0.40	-1.9	-21
180	242	266	252	256	20.4	246.2	190	1	191.0	14,800	5	0.5895	189.0	-1.40	-0.6	-10
180	215	239	225	256	20.5	219.1	168	2.5	170.5	14,800	5	0.5895	168.2	-0.99	-1.3	-19
180	188	211	198	255	19.5	192.1	147.5	3	150.5	14,800	5	0.5895	147.5	-0.68	-2.4	-31
180	157	180	166	257	20.2	160.3	123.5	2.5	126.0	14,800	5	0.5895	123.1	-0.40	-2.5	-28

Figure 11.9 Selecting Data to be Graphed

The next step is select the “Insert” tab which is seen in the top left hand corner of Fig 11.9. The center section of the Insert tab is a “Chart” menu. For this example and 99% of other flight test data, you will use the " Scatter" chart type. Initially, the "Line" type looks appropriate, but the x-axis in the Line type is not a continuous variable. Use the Scatter type. There are five different subtypes as shown below; select the one which does not include lines. We'll add a curve fit later.

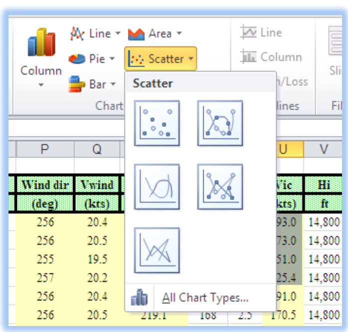


Figure 11.10 Scatter Chart Sub-Types

Once you’ve selected a sub-type, Excel creates the chart with default settings as shown in Figure 11.11.

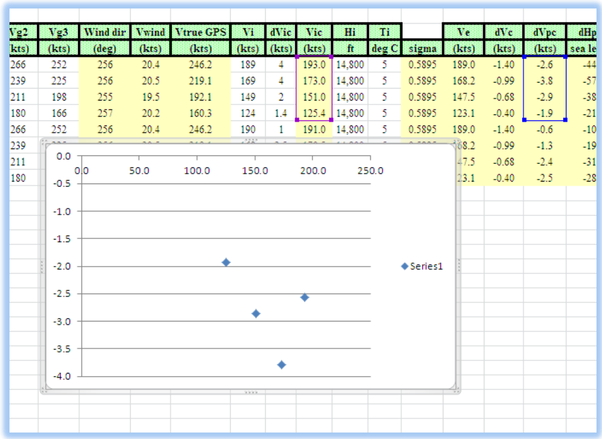


Figure 11.11 Chart with Default Settings

10.3.2 Modifying the Chart

The remainder of our effort to create the desired Chart is simply to modify the default settings to our specific desired formats. One way to start that process is to select the "Quick Layout" icon on the "Chart Tools – Design" tab, as shown in figure 11.12.

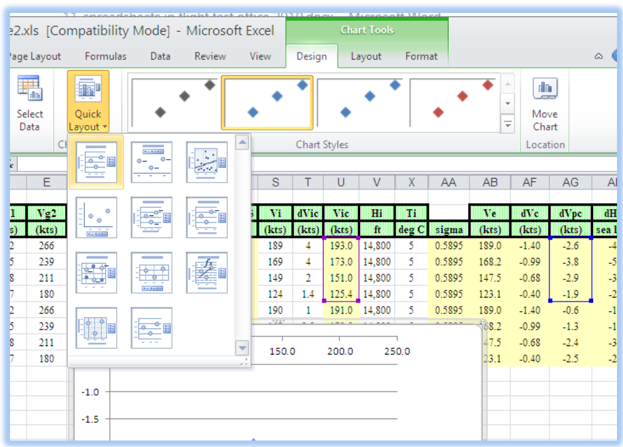


Figure 11.12 Chart Quick Layout Formatting.

Choosing the top left choice will change the selected chart to include a title, axis labels and a legend, as shown next:

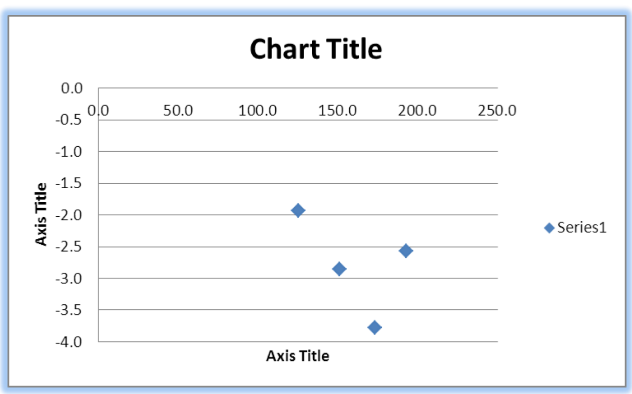


Figure 11.13 Quick Layout 1

Now clicking on the placeholders for the titles, allows easy modification as shown next:

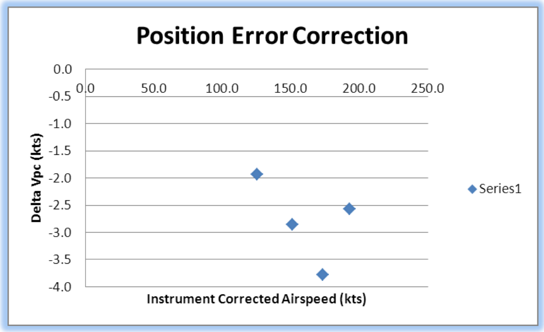


Figure 11.14 Titles Added

If you hover the mouse pointer over an element of the chart (such as the horizontal axis) an indicator will appear showing what you are over. Then right-clicking on that element will generate a pop-up menu that will allow you to format that element. These two steps are shown below:

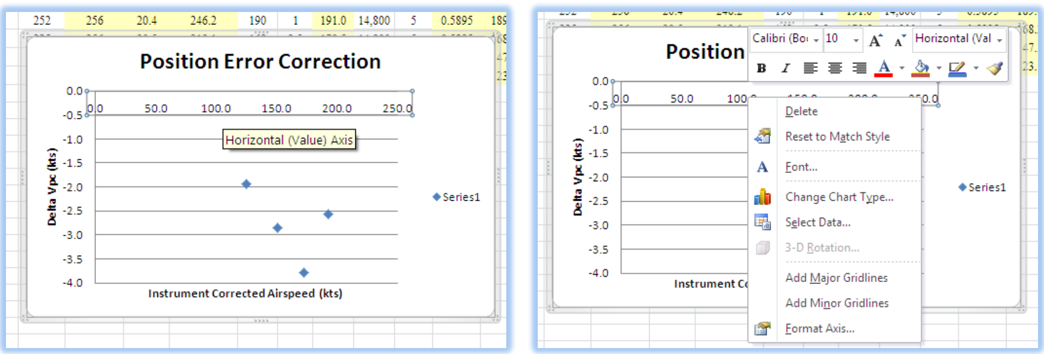


Figure 11.15 Pop-up Menu for Chart Horizontal Axis

From that pop-up menu select "Add Major Gridlines" and then repeat the hover/right-click to allow selection of "Format Axis . . ." which gives the following dialog box:

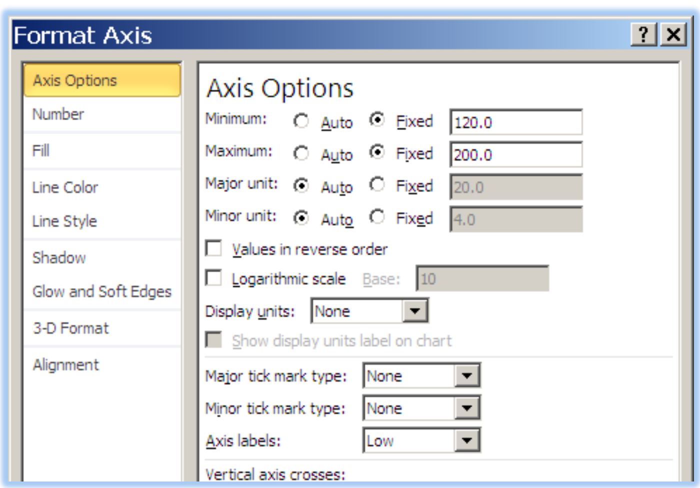


Figure 11.16 Format Axis Dialog

In this dialog box you can change the axis min and max values from the default to a scale that suits the purpose better. You can also change the x-axis labels to be low on the chart. By selecting "number" on the left you can choose the desired number of decimals or other formatting choices for the axis labels. Some of those changes can be seen in the next figure:

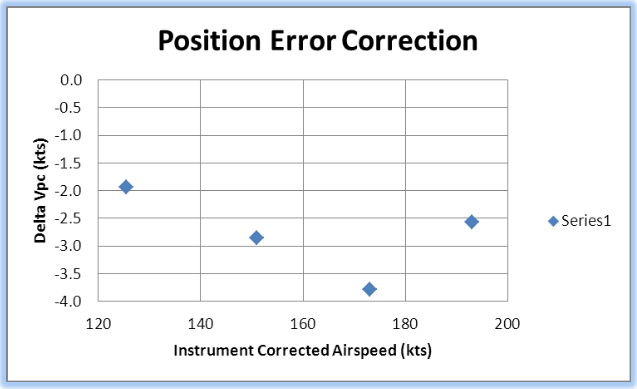


Figure 11.17 Interim Changes

Making similar changes to the vertical or y-axis results in the following chart depiction:

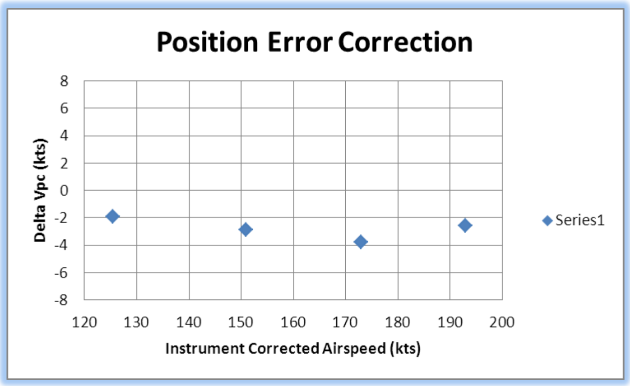


Figure 11.18 Modified Vertical Axis

Next we want to add a line or curve through the data. Be careful not to choose a line that connects the data "dots." It is unlikely that any of the data is exactly on the true curve as all flight test data has some error or uncertainty. Drawing a curve to *fit* the data is done by right-clicking on the data which generates a pop-up menu that allows selection of adding a "trendline" as shown next:

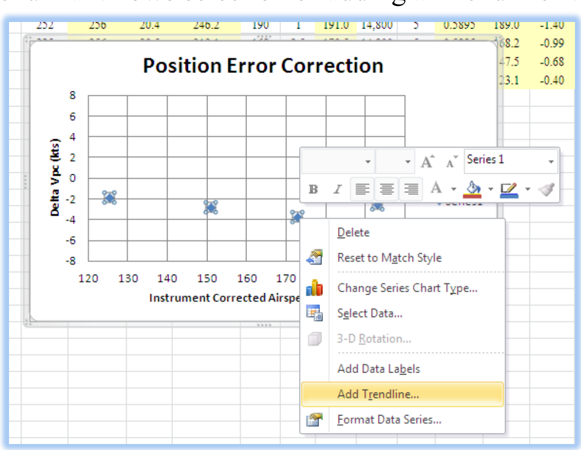


Figure 11.19 Adding a Curve Fit or Trendline

The pop-up menu includes the options shown in the following dialog box. Normally, at most a second order curve fit should be used unless there are technical reasons for choosing a more complex curve fit.

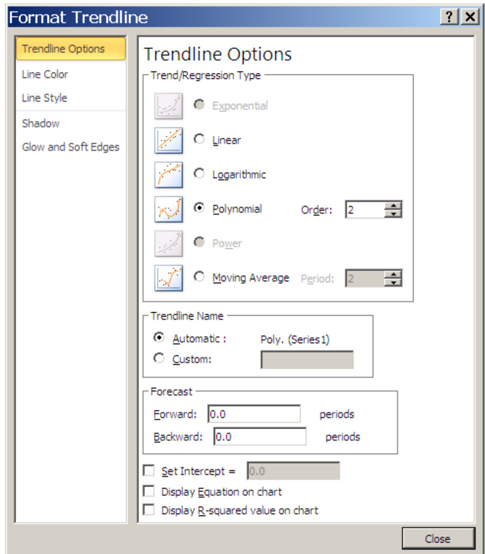


Figure 11.20 Formatting a Trendline

Looking back at Figure 11.7 we have two different groups of data with different curve fits: the *Copilot System* and the *Wingtip System*. To add the *Wingtip System* to our graph we switch back to the Data! worksheet and highlight the appropriate cells for the second system, Ul 1:14,AG11:14. Select Edit/Copy (<Ctrl C>). Then switch back to the graph. We want to paste the new data onto the same graph. But if we do the intuitive Edit/Paste (<Ctrl V>) we get a perhaps unexpected result, more blue diamonds. That is, more points in the existing series, as opposed to a new series, which is what we intended. The fix is to select "Edit/Paste Special...", making sure to select "Add new cells as" a "New series" and to check the box labeled "Categories (X Values) In First Column".

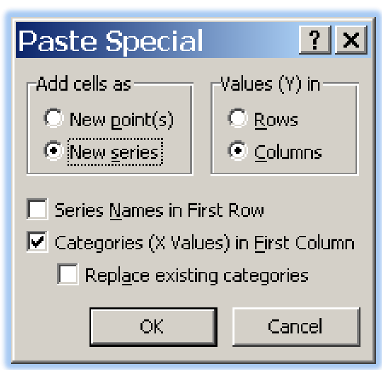


Figure 11.21 Adding a Second Data Series

The new series should be formatted as desired. Add a second order polynomial curve fit, formatting it as desired also.

Notice that the legend automatically includes four items: "Series 1, Series 2, Poly. (Series 1) and Poly. (Series 2)." To give the legend descriptive names (Copilot and Wingtip System), there are two ways (at least). One would be to right-click a data point and then choose “Select Chart Data” from the pop-up menu. Next select "Edit" on the new dialog box which will lead to the following new dialog box wherein you can type a descriptive name for the series as shown below.

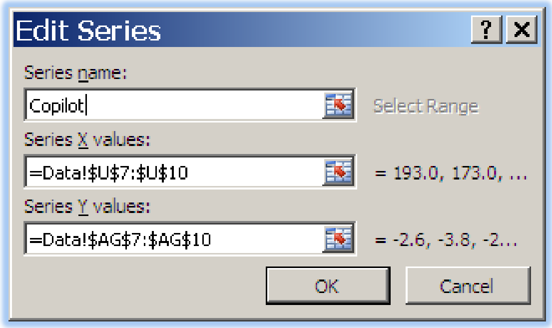


Figure 11.22 Naming a Series

An alternative would be to select the series by left-clicking on any data point in the series and look in the "Formula Bar" near the top of the sheet. The series formula has a place for the name of the series. The formula begins with "=SERIES(,Data!....)". Anything entered before that first comma will be treated as the series name. It must be entered within quotation marks, however, as shown below.

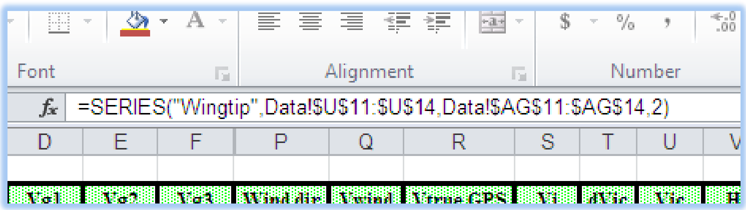


Figure 11.23 Series Formula Definition

The legend can be improved by deleting the entry for the curve fits as they are normally not needed. After selecting the legend, left-click on the entry for a polynomial and then hit the delete key on the keyboard. Afterwards you can right-click on the legend and select "Format Legend" from the pop-up menu. In the figure below a white fill was added along with a black border. After dragging the legend to the upper right part of the chart, the main graph was selected and resized to fill the chart area better.

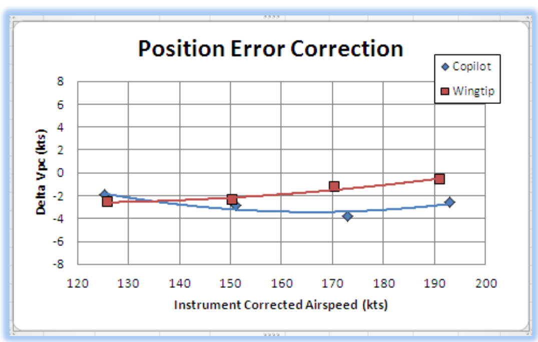


Figure 11.24 Legend Formatted

From a data analysis point of view it helps tremendously if you can graphically show the requirements as well as the test results. Notice that in Figure 11.7, the FAA limits for pitot-static errors are shown graphically along with the data for both pitot-static systems. To add these limits, we simply add another series, or in this case, two new series, one for the upper limit and one for the lower limit. In the example Excel workbook there is a separate spreadsheet with the limits. Select the positive limits and "Edit/Paste Special..." them onto the graph. Then select the negative limits and paste them the same way.

Once they are on the graph as new series format them a little differently than the previous series. The first difference is that you should select "None" for the Marker. Next under Line, select a red "Color", a heavy "Width" and a dashed "Style." Do this the same for both upper and lower limits. As previously done, change one of the two to have a series name of "FAA Limits" for the legend.

On most flight test charts you need to add information about the test such as aircraft type, date, configuration, etc. This information is normally put in a text box. To do this select “Text Box” from the “Insert Tab” at the top of the spreadsheet as shown below and then start typing. Later you can right-click the text box and format it as desired.

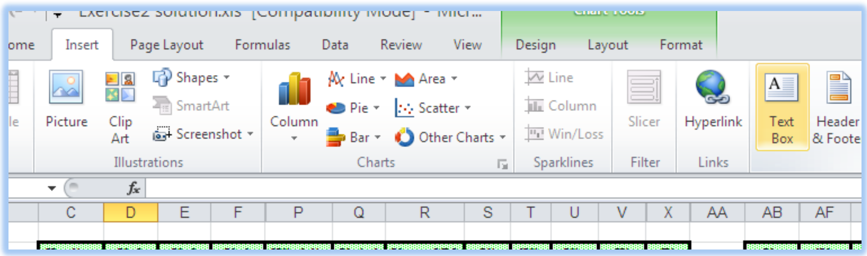


Figure 11.25 Inserting a Text Box

The very last thing to do may be to relocate the chart as a separate sheet in the Excel workbook. To do this, right-click on the chart and select “Move Chart.” In the dialog box you may select as a new sheet. Normally it would be fine to have interim, working graphs embedded in a spread sheet, but final report charts are probably better placed as separate sheets in the workbook.

10.3.3 Exercise 2

Given a spreadsheet with the data used above, go through each step and create the desired graph to look like Figure 11.7.

10.4 Other Topics

10.4.1 Cell Names

As we saw back in the first section of these notes, copying and pasting equations results in a relative addressing, which is usually just what we want. Sometimes, however, this can lead to unwanted results. Consider the situation in Figure11.26.

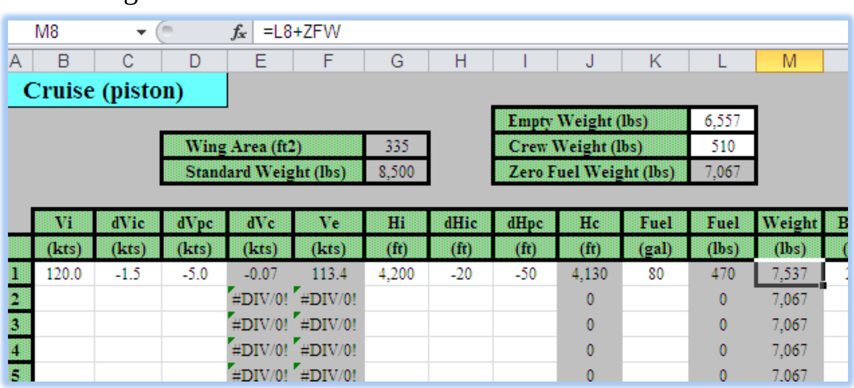


Figure 11.26 Use of Cell Names

Cell M8 is intended to be the weight of the aircraft at the time the data point was taken. The speed was 120 kts and the fuel onboard at that point was 80 gallons. So in cell M8 we add the weight of the fuel to the zero fuel weight (ZFW), which is calculated in cell L4. But if we do "=L8 + L4" we can't copy and paste for additional data points, because then cell M9 would be "=L9 + L5" and L5 is not the ZFW.

There are two solutions to this problem. Number one is to use symbols that tell Excel that you want to use absolute addressing instead of the default relative addressing. The use of"$" before either or both of the column letter or row number tells Excel that when it is copied and pasted it will remain constant. Thus "=L8 + L4" will copy and paste down one cell to be "=L9 + L4" preserving the intention to add the current fuel to the ZFW.

The second solution is somewhat better in that the equation will be easier to read and understand. If you give cell L4 a name, such as "ZFW" then you can refer to the name in equations as shown in Figure 11.27. Whenever you use ZFW in an equation it will use the contents of the cell to which the name refers, in this case cell L4. There are two ways to name a cell. The simple way is to select the cell and type in the "name box" shown in figure 11.27. Another way is to choose "Define Name" or "Name Manager" on the Formulas tab.

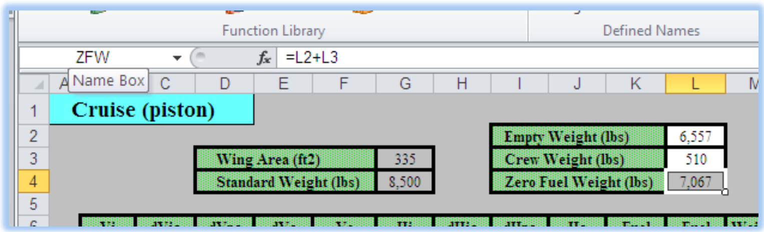


Figure 11.27 Name Box

10.4.2 Hiding Cells

Intermediate engineering calculations are frequently done best in small increments. This aids in analyzing errors in equations and formulae but potentially spreads the results out over several pages requiring the user to page through lots of unnecessary data to get to the final answer. Worse, it can mask relationships because the user can't see both input and output at the same time. It would be best to show input and output side by side on the same page. An easy way to do this is to hide the intermediate steps as shown in Figure 11.28.

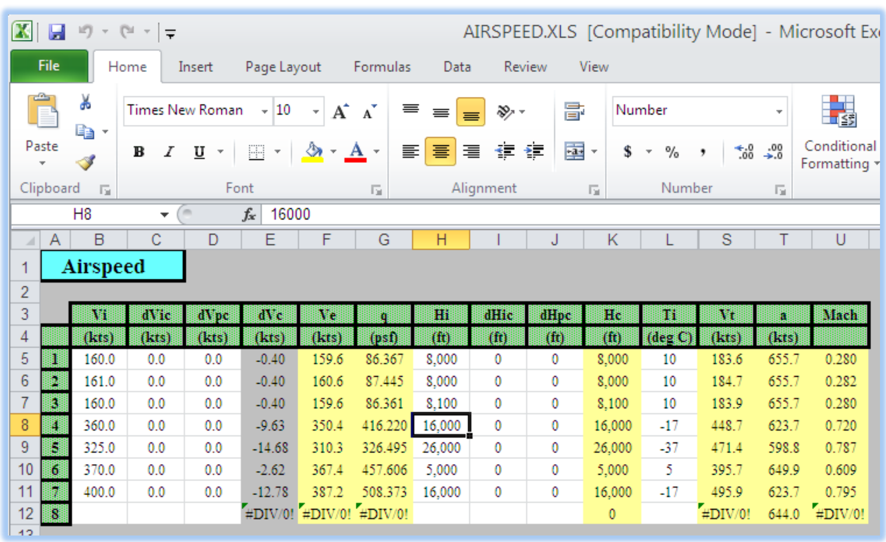


Figure 11.28 Hidden Columns

Notice that the columns labels skip from L to S. The six missing columns are intentionally hidden as they contain six intermediate calculations (delta, theta, sigma, q/p, etc) used in determining the true airspeed and the speed of sound, a. Displaying the intermediate calculations adds nothing to the user's value and would just make it difficult to see the relationship between input data (speed, altitude, and temperature) and output (true airspeed and Mach number).

To hide columns, select them by left-clicking the desired letter (and dragging to obtain multiple columns if needed), and then right-click the selection to "Hide" the column(s).

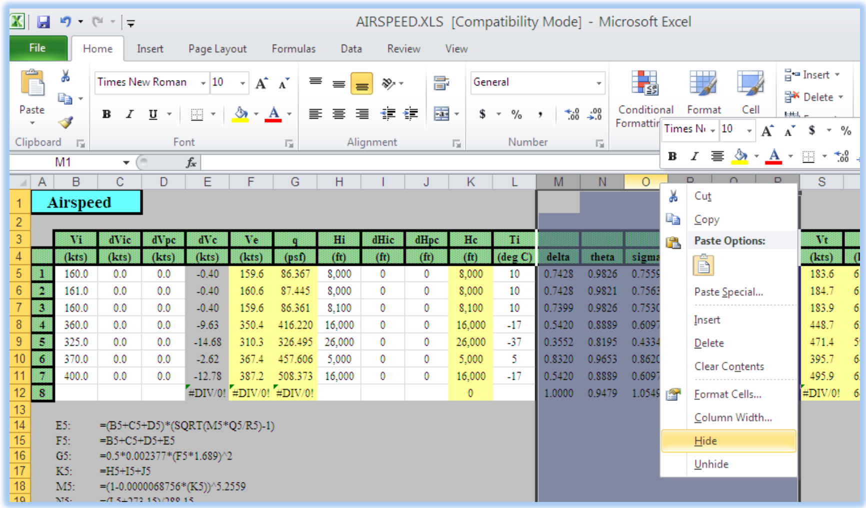


Figure 11.29 Hiding Columns

To unhide columns previously hidden, select the two visible columns on either side of the hidden area and select "Unhide." In the above examples, the hidden columns are important for getting correct results, so replicating the hidden portion is important when copying and pasting. If the selected cell range during a copy operation spans a group of hidden cells the hidden cells will be also copied and subsequently pasted.

10.4.3 Functions

Excel comes with many built in functions. Examples include the previously referred to square root function, but there are literally hundreds of others available. Remembering their names and argument types would be impossible (is the sine function spelled SIN() or SINE(), and does it take radians or degrees?). One solution is to invoke the "Help" menu (shortcut <F1> or click on the blue circled question mark) to look up a particular function. An alternative is the "Formulas/Insert Function..." icon which brings up a dialog box as shown in Figure 11.30.

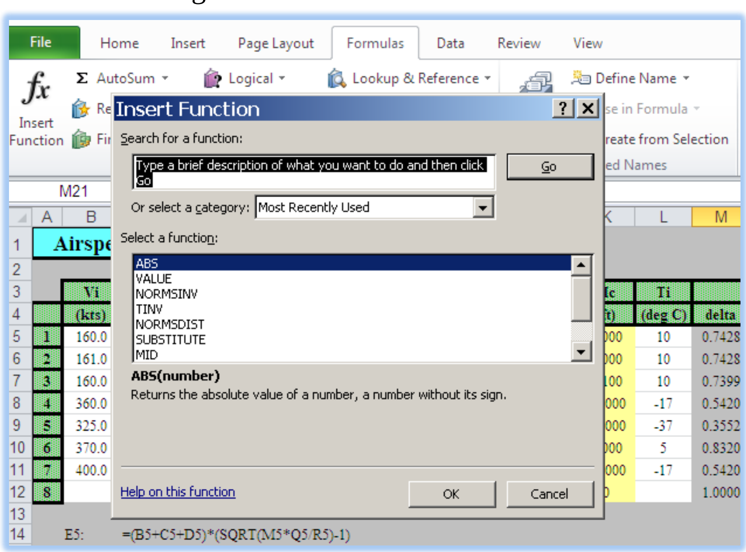


Figure 11.30 Inserting a Function into an Equation

With the dialog box shown above, you can choose from many different categories of functions making it easier to find what you need. Also the bottom of the dialog box may be enough to tell you what you need to know about the function.

As an example of the utility of these functions, consider the ratio of pressure at altitude to the pressure at sea level on a standard day. This pressure ratio is used repeatedly in flight test data reduction. The equation will be developed in a later class on Standard Atmosphere, but for now take the following on faith. For altitudes below 36,089 feet, the pressure ratio is:

$$\delta = P/Po = (1 - 6.8756 * 10^{-6} * H)^{5.2559}$$

Above 36,089 feet the equation changes to:

$$\delta = 0.223358 * e^{-0.00004806 * (H - 36089)}$$

In Excel you can use a logical IF function along with MATH functions to determine the pressure ratio for a given altitude. The IF function works as follows:

=IF (logical test, result if true, result if false)

C4 fx =IF(A4<36089,(1-0.0000068756*(A4))^5.2559,0.223358*EXP(-0.0000480614*(A4-36089)))

A	B	C	D	E	F	G	H	I	J	K
H (Kft)	**T (deg K)**	**delta**	**theta**	**sigma**						
0	288.2	1.0000	1.0000	1.0000						
1,000	286.2	0.9644	0.9931	0.9711						
2,000	284.2	0.9298	0.9862	0.9428						
3,000	282.2	0.8962	0.9794	0.9151						
4,000	280.2	0.8637	0.9725	0.8881						
5,000	278.2	0.8320	0.9656	0.8617						

Figure 11.31 Example Use of Functions

10.4.4 Add-ins

A problem with using complicated functions as shown above is that you have to type them in to every spreadsheet that will use them. An alternative to this is to make a brand new, user defined function for the pressure ratio. In order to use a "User Defined" function, you must either write your own using Visual Basic, or you could import a group of user-defined functions in the form of an Excel "Add-in." At NTPS, several common aerodynamic functions have been written and can be added to your system. The good part is that it makes available to all of your spreadsheets a set of debugged functions that could be very time consuming for you to individually generate in your own spreadsheets. Complicated functions can also lead to errors that are difficult to find and correct. The downside is that the spreadsheets using the add-in are now dependent on having the add-in installed on the machine. If you move the spreadsheet to another computer it won't work unless the same add-in is also installed on the new computer in the same location.

The NTPS flight test add-in is a file titled "flight_test.xla". To work properly, the standard location to copy the file on the host computer should be:

C :\WINDOWS\Application Data\Microsoft\AddIns

After copying the file to the above location, the add-in must be "installed" before you can use the functions. To install the add-in, select “Options” from the “File” tab and then select “Add-Ins.” Then click on the “Go” button to “Manage Excel Add-ins.” There you can check that available add-ins are selected or if the user defined flight test add-ins are not available, hit the “Browse..” button and navigate to the C:\WINDOWS\Application Data\Microsoft\AddIns directory, select the flight_test.xla file and hit <OK>. Now the flight test user defined functions should be available to your spreadsheets. As an example, if you need the previously mentioned pressure ratio, all you need to do is as shown below.

C4 fx =pressratio(A4)

	A	B	C	D	E
1	**H (Kft)**	**T (deg K)**	**delta**	**theta**	**sigma**
2	0	288.2	1.0000	1.0000	1.0000
3	1,000	286.2	0.9644	0.9931	0.9711
4	2,000	284.2	0.9298	0.9862	0.9428
5	3,000	282.2	0.8962	0.9794	0.9151
6	4,000	280.2	0.8637	0.9725	0.8881

Figure 11.32 User Defined Functions

The user defined functions are written in Visual Basic for Applications. The functions can be viewed or even modified if you are familiar with Visual Basic. The functions provided currently are shown in the following table:

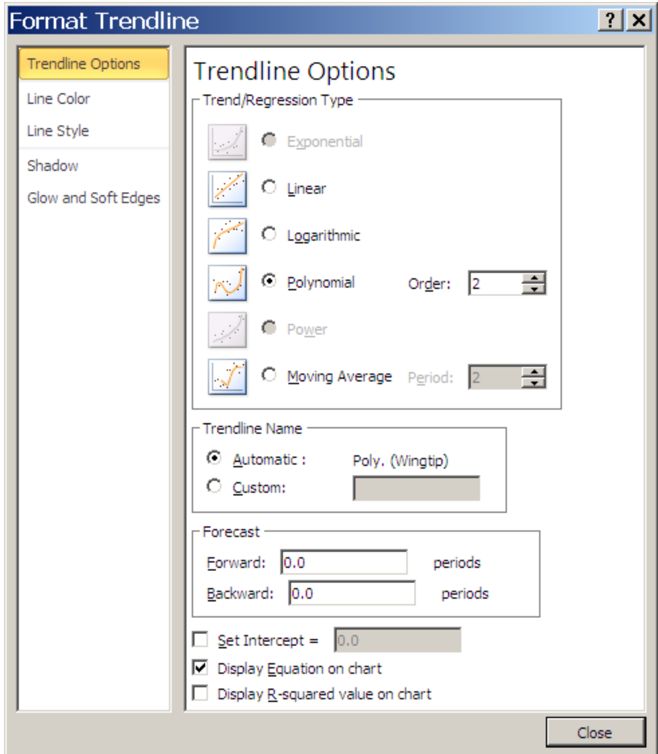
Function Name	Units	Arguments
dVc	knots	altitude in feet calibrated airspeed in knots
Mach	-	altitude in feet calibrated airspeed in knots
InvMach	knots	altitude in feet non-dimensional Mach number
PressRatio	-	altitude in feet
InvPressRatio	ft	pressure ratio
Sigma	-	altitude in feet temperature in degrees Kelvin
TempStd	degK	altitude in feet
Vtrue	knots	altitude in feet temperature in degrees Kelvin calibrated airspeed in knots

Table 11.1 Flight Test Add-In Functions

10.4.5 Curve Fit Equations

Frequently in engineering analysis we would like to use the equation of a best curve fit. As an example, theory says that in an aircraft's drag polar, plotting lift coefficient squared vs. the drag coefficient should result in a straight line. If we do a first order best curve fit in Excel using C_L^2 vs C_D we could use the equation of the resulting curve fit to "draw" the second order polar curve fit on a plot of C_L vs C_D. Similarly, getting the slope of a 3rd order curve fit of time vs. true airspeed gives the instantaneous acceleration which can be used to determine the potential rate of climb at that speed.

In order to do either of the above we need the equation that Excel obviously knows because it solved for the equation in order to determine the curve fit. All we need to do is to get Excel to display it. If you have added a "trendline" to a series of data, you can also display the curve fit by right-clicking on it to get the popup menu and selecting "Format Trendline...". On the resulting dialog box select the Options tab, where you can then check the "Display equation on chart" box as shown next:



The equation will now be displayed on the graph itself. Occasionally, you will need to change the format of the equation. To do so, select it, right-click on it, and then format it as needed to include fills and borders. An engineering reason to format it would be to set the number of significant figures displayed. For very small or large numbers Excel defaults to scientific notation, which is fine, but only one significant figure is shown, which may not be fine. By formatting the equation and then selecting the "number" tab, you can set the number format to scientific with two decimal places.